THE CHEVALLEY INVOLUTION AND A DUALITY OF WEIGHT VARIETIES

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To the memory of Armand Borel

Abstract. In this paper we show that the classical notion of association of projective point sets, [DO], Chapter III, is a special case of a general duality between weight varieties (i.e. torus quotients of flag manifolds) of a reductive group G induced by the action of the Chevalley involution on the set of these quotients. We compute the dualities explicitly on both the classical and quantum levels for the case of the weight varieties associated to $GL_n(\mathbb{C})$. In particular we obtain the following formula as a special case. Let $\mathbf{r} = (r_1, \ldots, r_n)$ be an n-tuple of positive real numbers and $M_{\mathbf{r}}(\mathbb{CP}^m)$ be the moduli space of semistable weighted (by $\mathbf{r}$) configurations of n points in $\mathbb{CP}^m$ modulo projective equivalence, see for example [FM]. Let $\mathbf{\Lambda}$ be the vector in $\mathbb{R}^n$ with all components equal to $\sum_i r_i/(m+1)$. Then $M_{\mathbf{r}}(\mathbb{CP}^m) \cong M_{\mathbf{\Lambda}-\mathbf{r}}(\mathbb{CP}^{n-m-2})$ (the meaning of $\cong$ depends on $\mathbf{r}$ and will be explained below, see Theorem 1.6). We conclude by studying "self-duality" i.e. those cases where the duality isomorphism carries the torus quotient into itself. We characterize when such a self-duality is trivial, i.e. equal to the identity map. In particular we show that all self-dualities are nontrival for the weight varieties associated to the exceptional groups. The quantum version of this problem, i.e. determining for which self-adjoint representations V of G the Chevalley involution acts as a scalar on the zero weight space $V[0]$, is important in connection with the irreducibility of the representations of Artin groups of Lie type which are obtained as the monodromy of the Casimir connection, see [MTL], and will be treated in [HMTL].

1. Introduction

In this paper we show that the classical notion of association of projective point sets,[DO], Chapter III, is a special case of a general duality between weight varieties (i.e torus quotients of flag manifolds) of a reductive group G induced by the action of the Chevalley involution on the set of these quotients. We will build up the theory in stages, first the duality for Grassmannians of $GL_n(\mathbb{C})$, then for general flag manifolds of $GL_n(\mathbb{C})$ then the duality for general flag manifolds of general semisimple complex groups. At each stage there are three types of isomorphism theorems, the first type is a Kähler isomorphism of symplectic quotients, the second type is an algebraic isomorphism of Mumford quotients and the third type is an explicit formula for the isomorphism of homogeneous coordinate rings in terms of combinatorial Lie theory. We now give details.

1.1. Duality results for torus quotients of Grassmannians.

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1.1.1. *Duality of symplectic quotients of Grassmannians.* We describe the duality result for symplectic quotients of Grassmannians. Let B be the nondegenerate symmetric bilinear form on $\mathbb{C}^n$ such that the standard basis $\{\epsilon_1, ..., \epsilon_n\}$ is orthonormal. Let T be the maximal compact subgroup of H (so T is a product of $\dim_{\mathbb{C}}(H)$ circles). Then the operation Ψ of taking orthogonal complement with respect to B induces a map $\Psi : Gr_k(\mathbb{C}^n) \to Gr_{n-k}(\mathbb{C}^n)$ which carries torus orbits to torus orbits, since it satisfies the formula

$$\Psi(h \cdot x) = h^{-1} \cdot \Psi(x) \tag{1}$$

Let $|\mathbf{r}| = \sum_{i=1}^n r_i$ and $a = |\mathbf{r}|/k$. We will see in §2 that Ψ maps the torus momentum level $\mathbf{r} = (r_1, ..., r_n)$ for T to the torus momentum level $\mathbf{\Lambda} - \mathbf{r} = (a - r_1, ..., a - r_n)$. Consequently we obtain the following duality theorem.

Theorem 1.1. *Let $\mathbf{r} = (r_1, ..., r_n) \in \mathfrak{t}^* \cong \mathbb{R}^n$ be in the image of the momentum mapping for the action of T on $(Gr_k(\mathbb{C}^n), a\omega_k)$. Then the map $\Psi : Gr_k(\mathbb{C}^n) \to Gr_{n-k}(\mathbb{C}^n)$ induces a homeomorphism (also to be denoted Ψ) of the symplectic quotients $(Gr_k(\mathbb{C}^n), a\omega_k)//_{\mathbf{r}}T$ to $(Gr_{n-k}(\mathbb{C}^n), a\omega_{n-k})//_{\mathbf{\Lambda}-\mathbf{r}}T$. In case $\mathbf{r}$ is not on a wall, see* [FM], *§4, then both symplectic quotients are smooth and the map Ψ is a Kähler isomorphism.*

1.1.2. *Duality of Mumford quotients of Grassmannians.*
Linearization of the torus action. Let H be the maximal torus of $GL_n(\mathbb{C})$ consisting of the nonzero diagonal matrices and let PH be the image of H in $PGL_n(\mathbb{C})$. Let ω_k be the symplectic form on $Gr_k(\mathbb{C}^n)$ induced by the embedding into $\mathfrak{u}(n)^*$ as the orbit of the k–th fundamental weight ϖ_k, the highest weight of the representation of $GL_n(\mathbb{C})$ on $\bigwedge^k(\mathbb{C}^n)$. Let $\mathcal{L}_k$ be the dual of the k–th exterior power $\mathcal{T}_k$ of the tautological k-plane bundle over $Gr_k(\mathbb{C}^n)$. We will refer to $\mathcal{T}_k$ as the tautological line bundle over $Gr_k(\mathbb{C}^n)$. Let $P \subset GL_n(\mathbb{C})$ be the stabilizer of the coordinate plane spanned by the first k standard basis vectors. The bundle $\mathcal{L}_k$ is the homogeneous $GL_n(\mathbb{C})$–bundle over $Gr_k(\mathbb{C}^n) = GL_n(\mathbb{C})/P$ with the isotropy representation $det_k^{-1} : P \to \mathbb{C}^*$ where det_k assigns to $p \in P$ the determinant of the upper k by k block of p. In particular the total space of $\mathcal{L}_k$ is the quotient of the product $GL_n(\mathbb{C}) \times \mathbb{C}$ by the equivalence relation $(g, z) \sim (gp, det_k(p)z)$. Since $\mathcal{L}_k$ admits a $GL_n(\mathbb{C})$ action it admits an H action. Since the entries of $\mathbf{r}$ are integers we may identify $\mathbf{r}$ with the character $\chi_{\mathbf{r}}$ of H whose value at the diagonal matrix with entries $(z_1, ..., z_n)$ is $z_1^{r_1} \cdots z_n^{r_n}$. For any integer b we may use this character to twist the action of H on the line bundle $\mathcal{L}_k^{\otimes b}$. We will use the symbol $\mathcal{L}_k^{\otimes b}(\mathbf{r})$ to denote the H–line bundle $\mathcal{L}_k^{\otimes b}$ equipped with the twisted (by $\chi_{\mathbf{r}}$) H action. The group H acts on $Gr_k(\mathbb{C}^n)$ through the quotient PH. In what follows we will need conditions on $\mathbf{r}$ that are necessary and sufficient in order that the action of H on $\mathcal{L}_k^{\otimes b}(\mathbf{r})$ descends to an action of PH. Let $|\mathbf{r}| = \sum_i r_i$.

Lemma 1.2. *The induced action of H on $\mathcal{L}_k^{\otimes b}(\mathbf{r})$ descends to an action of PH if and only if $bk = |\mathbf{r}|$.*

Proof. Let $h = \mu I$ be a nonzero scalar matrix. Then for $[g, z]$ in the total space of $\mathcal{L}_k^{\otimes b}(\mathbf{r})$ we have

$$h[g, z] = [hg, \chi_{\mathbf{r}}(h)z] = [gh, \mu^{|\mathbf{r}|}z] = [g, det_k(h)^{-b}\mu^{|\mathbf{r}|}z] = [g, \mu^{-kb+|\mathbf{r}|}z].$$

Thus $h[g,z] = [g,z] \Leftrightarrow bk = |\mathbf{r}|$. $\square$

The reader will verify that the condition $bk = |\mathbf{r}|$ is necessary in order that there exist a nonzero section of $\mathcal{L}_k^{\otimes b}(\mathbf{r})$ that is invariant under the group of nonzero scalar matrices. Thus if it is not satisfied there will be no nonzero H–invariant sections of $\mathcal{L}_k^{\otimes b}(\mathbf{r})$. For this reason we assume that $|\mathbf{r}|$ is divisible by k and we will reserve the symbol a for the quotient $|\mathbf{r}|/k$. We will abbreviate $\mathcal{L}_k^{\otimes a}$ to $\mathcal{L}_k^a$ henceforth. Let $\mathbf{\Lambda} = (a, a, ..., a) \in \mathbb{Z}_+^n$.

Definition 1.3. *For any integral* $\mathbf{r}$ *satisfying the condition that* $|\mathbf{r}|$ *is divisible by* k *we will refer to the line bundle* $\mathcal{L}_k^a(\mathbf{r})$ *equipped with the previous action of* PH *as the* $\mathbf{r}$*–linearization of the action of* PH *on* $Gr_k(\mathbb{C}^n)$*.*

The bundle isomorphism induced by the complex Hodge star. Before stating our duality theorem concerning Mumford quotients we need to recall the definition of a semistable point. A point $x \in Gr_k(\mathbb{C}^n)$ is semistable for the $\mathbf{r}$–linearization of PH if there exists some $N \in \mathbb{Z}_+$ and an H–invariant section s of $\mathcal{L}_k^{\otimes Na}(N\mathbf{r})$ such that $s(x) \neq 0$ (here the symbol $\mathcal{L}(N\mathbf{r})$ means we twist the action of H on $\mathcal{L}$ by the character $\chi_{\mathbf{r}}^N$).

Suppose now that $\mathbf{r} = (r_1, ..., r_n) \in \mathbb{Z}_+^n$. We may form the Mumford quotients $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}H$ and $Gr_{n-k}(\mathbb{C}^n)//_{\Lambda-\mathbf{r}}H$ (using the linearizations corresponding to $\mathbf{r}$ and $\Lambda - \mathbf{r}$ respectively). We will prove these Mumford quotients are isomorphic varieties by constructing an explicit isomorphism of homogeneous coordinate rings. To this end choose a complex orientation of $\mathbb{C}^n$ and let $*$ be the complex Hodge star (see §2) associated to this orientation and the form B. We then define a bundle isomorphism $\hat{\Psi} : \mathcal{L}_k \to \mathcal{L}_{n-k}$ as follows. Let $x \in Gr_k(\mathbb{C}^n)$. Let $\tau \in \bigwedge^k((\mathbb{C}^n)^*)$ and let $res_x : \bigwedge^k((\mathbb{C}^n)^*) \to \bigwedge^k(x^*)$ be the restriction map. Then we define $\hat{\Psi}$ by

$$\hat{\Psi}(res_x(\tau)) = res_{\Psi(x)}(*\tau).$$

It will be proved in §2.2 that $\hat{\Psi}$ is well defined and it will be proved in §4.3 that $\hat{\Psi}$ is an H–morphism of line bundles (with the H action on the target inverted). The bundle map $\hat{\Psi}$ induces a map of sections $\widetilde{\Psi} : \Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}_k) \to \Gamma(Gr_{n-k}(\mathbb{C}^n), \mathcal{L}_{n-k})$ given by

$$\widetilde{\Psi}(s)(x) = \hat{\Psi}(s(\Psi^{-1}(x))).$$

We will also use $\hat{\Psi}$ resp. $\widetilde{\Psi}(x)$ to denote the maps on tensor powers that are induced by $\hat{\Psi}$ resp. $\widetilde{\Psi}$. We will prove in §4.3 that $\widetilde{\Psi}$ carries H–invariants to H–invariants and consequently induces an isomorphism of homogeneous coordinate rings. We summarize these statements in

Theorem 1.4.

(1) *For all positive integers* N *the bundle isomorphism* $\hat{\Psi}$ *from* $\mathcal{L}_k^{\otimes Na}(N\mathbf{r})$ *to* $\mathcal{L}_{n-k}^{\otimes Na}(N(\mathbf{\Lambda} - \mathbf{r}))$ *satisfies*

$$\hat{\Psi} \circ h = h^{-1} \circ \hat{\Psi}, h \in H.$$

(2) *The induced isomorphisms of sections* $\widetilde{\Psi}$ *carry* H*–invariant sections of* $\mathcal{L}_k^{\otimes Na}(N\mathbf{r})$ *to* H*–invariant sections of* $\mathcal{L}_{n-k}^{\otimes Na}(N(\mathbf{\Lambda} - \mathbf{r}))$ *for all* N*, and consequently induce an isomorphism again to be denoted* $\widetilde{\Psi}$ *of the homogeneous coordinate rings of the Mumford quotients* $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}H$ *and* $Gr_{n-k}(\mathbb{C}^n)//_{\mathbf{\Lambda}-\mathbf{r}}H$*.*

Remark 1.5. *We could have avoided the choice of form B by defining the duality map to be the map from the k–planes in $\mathbb{C}^n$ to the $(n-k)$–planes in $(\mathbb{C}^n)^*$ given by mapping a plane x to its annihilator as suggested by* [Do], *Exercise 12.7, page 203. However then we would not have had the explicit bundle map $\widetilde{\Psi}$ induced by the complex Hodge star.*

We will not give an explicit formula for the isomorphism $\widetilde{\Psi}$ on the standard basis for the homogeneous coordinates because it is not much simpler than the more general formula of Theorem 1.12. Moreover the formulas in this case may be found in [DO], Ch. III.

1.2. **Duality of weighted projective configurations.** We now explain how the duality theorem for weighted projective configurations (and its special case , the association of projective point sets) follows from the duality of symplectic torus quotients of Grassmannians.

Theorem 1.6. *Let $M_{\mathbf{r}}(\mathbb{CP}^m)$ be the moduli space of semistable weighted (by $\mathbf{r}$) configurations of n–points on $\mathbb{CP}^m$. Then*

$$M_{\mathbf{r}}(\mathbb{CP}^m) \cong M_{\mathbf{\Lambda}-\mathbf{r}}(\mathbb{CP}^{n-m-2}).$$

By the symbol $\cong$ we mean the two spaces are homeomorphic and in case $\mathbf{r}$ is not on a wall, ([FM], *§4),then they are isomorphic as Kähler manifolds. If $\mathbf{r}$ is integral then by $\cong$ we mean isomorphism as algebraic varieties.*

Proof. In what follows the symbol $\cong$ will have the same meaning as in the statement of the Theorem. By Gelfand-MacPherson duality, see for example [FM], §8, we have $M_{\mathbf{r}}(\mathbb{CP}^m) \cong Gr_{m+1}(\mathbb{C}^n)//_{\mathbf{r}}H$ and $M_{\mathbf{\Lambda}-\mathbf{r}}(\mathbb{CP}^{n-m-2}) \cong Gr_{n-m-1}(\mathbb{C}^n)//_{\mathbf{\Lambda}-\mathbf{r}}H$. But by Theorem 1.1 we have $Gr_{m+1}(\mathbb{C}^n)//_{\mathbf{r}}H \cong Gr_{n-m-1}(\mathbb{C}^n)//_{\mathbf{\Lambda}-\mathbf{r}}H$. □

Remark 1.7. *If all the components of $\mathbf{r}$ are equal (the "democratic linearization") then the resulting isomorphism is the classical association isomorphism, see* [DO], *Ch. III.*

1.3. **Duality results for torus quotients of flag manifolds.** It is remarkable that the duality theorems for Mumford quotients and symplectic quotients for Grassmannians almost immediately imply the corresponding results for torus quotients of general flag manifolds. The proofs are based on using the following diagram to promote the duality theorem for Grassmannians to a duality theorem for flag manifolds. We let $\Psi : F_{\mathbf{k}}(\mathbb{C}^n) \to F_{\mathbf{l}}(\mathbb{C}^n)$ be the mapping that makes the following diagram commute.

$$\begin{tikzcd} F_{\mathbf{k}}(\mathbb{C}^n) \arrow[r, "\Psi"] \arrow[d, "i"'] & F_{\mathbf{l}}(\mathbb{C}^n) \arrow[d, "i"] \\ \prod_{i\leq m} Gr_{k_i}(\mathbb{C}^n) \arrow[r, "F\circ\prod_i \Psi_i"] & \prod_{i\leq m} Gr_{l_i}(\mathbb{C}^n) \end{tikzcd}$$

Here $\mathbf{k} = (k_1, ..., k_m)$ resp. $\mathbf{l} = (l_1, ..., l_m) := (n-k_m, ..., n-k_1)$ and $F_{\mathbf{k}}(\mathbb{C}^n)$ resp. $F_{\mathbf{l}}(\mathbb{C}^n)$ denotes the manifold of flags consisting of subspaces of dimensions $k_1 < \cdots k_m$ resp. $l_1 < \cdots < l_m$ and F is the map on products of Grassmannians that reverses the order of the factors. We assume $F_{\mathbf{k}}(\mathbb{C}^n)$ is given the symplectic structure inherited by embedding it as the coadjoint orbit of $\lambda = a_1\varpi_{k_1} + \cdots a_m\varpi_{k_m}$.

We put $\mathbf{a} = (a_1, ..., a_m)$, $\mathbf{b} = (b_1, ..., b_m) := (a_m, ..., a_1)$ and $|\mathbf{a}| = \sum a_i$. We let $\mathbf{\Lambda} = (|\mathbf{a}|, \cdots, |\mathbf{a}|)$ and $\mathbf{s} = \mathbf{\Lambda} - \mathbf{r}$.

1.3.1. *The duality theorem for symplectic quotients of flag manifolds.* First the duality theorem for symplectic quotients (the proof may be found in §2).

Theorem 1.8. *The map Ψ induces a homeomorphism of symplectic quotients:*

$$\overline{\Psi} : F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}T \to F_{\mathbf{l}}(\mathbb{C}^n)//_{\mathbf{s}}T.$$

Furthermore, if $\mathbf{r}$ is a regular value of the momentum mapping, then the symplectic quotients are Kähler manifolds and $\overline{\Psi}$ is a Kähler isomorphism.

Now the duality theorem for Mumford quotients. Let $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ be the homogeneous line bundle with isotropy representation the character corresponding to the negative of the dominant weight $\lambda = a_1\varpi_{k_1} + \cdots + a_m\varpi_{k_m}$. We let $\mathbf{r} \in \mathbb{Z}_+^n$ and let $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}(\mathbf{r})$ be the H-bundle represented by the line bundle $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ with the H action twisted by the character corresponding to $\mathbf{r}$. We first give the relation between $\mathbf{a}$ and $\mathbf{r}$ that is necessary and sufficient in order that PH act on $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}(\mathbf{r})$. The following lemma is Lemma 5.2 in the text.

Lemma 1.9. *The action of H on $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}(\mathbf{r})$ descends to an action of PH iff $|\mathbf{r}| = \sum_i a_i k_i$.*

We define the bundle map $\hat{\Psi}$ to be the tensor product of the bundle maps for the Grassmannians followed by a reversal of tensor factors.

We now state our isomorphism theorem for Mumford quotients, the proof is to be found in §4.

Theorem 1.10. (1) *The map $\widetilde{\Psi}$ induces an isomorphism of graded rings:*

$$\bigoplus_{N=0}^{\infty} \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{N\mathbf{a}}(N\mathbf{r}))^H \cong \bigoplus_{N=0}^{\infty} \Gamma(F_{\mathbf{l}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{l}}^{N\mathbf{b}}(N\mathbf{s}))^H.$$

(2) *Equivalently, the map $\widetilde{\Psi}$ induces an isomorphism of Mumford quotients:*

$$F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H \cong F_{\mathbf{l}}(\mathbb{C}^n)//_{\mathbf{s}}H.$$

An explicit formula for the ring isomorphism $\widetilde{\Psi}$. We next give an explicit formula for $\widetilde{\Psi}$ on the homogeneous coordinate ring of the flag manifold in terms of semistandard Young tableaux. The proof of the following theorem is to be found in §5.

Let $\lambda = \sum_i a_i\varpi_k$. The N-th graded summand $R_{\mathbf{k}}^{(N)}$ of the homogeneous coordinate ring of $F_{\mathbf{k}}(\mathbb{C}^n)$ is given by

$$R_{\mathbf{k}}^{(N)} = \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), (\mathcal{L}_{\mathbf{k}}^{\mathbf{a}})^{\otimes N}) = V_{N\lambda}.$$

Furthermore the N–th graded summand $(R_{\mathbf{k}}^{(N)})^H$ of the homogeneous coordinate ring of the Mumford quotient $F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H$ is given by

$$(R_{\mathbf{k}}^{(N)})^H = \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), (\mathcal{L}_{\mathbf{k}}^{\mathbf{a}})^{\otimes N}(N\mathbf{r}))^H = V_{N\lambda}(N\mathbf{r}).$$

The last symbol denotes the $N\mathbf{r}$–th weight space of the irreducible representation of $GL(n, \mathbb{C})$ with highest weight $N\lambda$. It is a standard result in representation theory [Bo], Ch. V, Theorem 5.3, that there is a basis for $V_{N\lambda}$ resp. $V_{N\lambda}(N\mathbf{r})$

consisting of the semistandard fillings resp. semistandard fillings of weight $N\mathbf{r}$ of the Young diagram $D_{\mathbf{a}}$ (the i^{th} row has length equal to the i^{th} component of $N\lambda$) by the integers between 1 and n inclusive. We will call this the standard basis of $R_{\mathbf{k}}^{(N)}$) (resp. $(R_{\mathbf{k}}^{(N)})^H$). If T is a semistandard filling (of any weight) of the above Young diagram by $1, 2, .., n$ we will let f_T denote the corresponding element of the homogeneous coordinate ring of the flag manifold $F_{\mathbf{k}}(\mathbb{C}^n)$. The set of all such f_T is a basis for $R_{\mathbf{k}}^{(N)}$. Note that $\widetilde{\Psi}$ induces a map from $R_{\mathbf{k}}^{(N)}$ to $R_{\mathbf{l}}^{(N)} = \Gamma(F_{\mathbf{l}}(\mathbb{C}^n), (\mathcal{L}_{\mathbf{l}}^{\mathbf{b}})^{\otimes N})$.

We will describe this map relative to the standard basis, a fortiori this will describe the map on the subrings of H–invariants.

We now describe a map from semistandard tableaux of weight $\mathbf{r}$ on the diagram $D_{\mathbf{a}}$ to semistandard tableaux of weight $\mathbf{\Lambda} - \mathbf{r}$ on the diagram $D_{\mathbf{b}}$ which we will denote $T \mapsto *T$. We explain how to obtain the dual tableau $*T$ with an example.

Example 1.11. *Let* $\lambda = 2\varpi_2 + \varpi_3, N = 1$.

$$T = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \Longrightarrow \widetilde{T} = \begin{array}{|c|c|c|} \hline 2 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & 2 & 1 \\ \hline 1 & 3 & 3 \\ \hline 4 & 5 & 4 \\ \hline \end{array} \Longrightarrow *T = \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 3 & 3 & 4 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

In general, extend T *to a rectangular* n *by* $|\mathbf{a}|$ *diagram and fill in the complementary indices in each column, listed in increasing order. Write the added columns in reverse order to get* $*T$.

We attach a sign ϵ_T to each semistandard tableau as follows. Form the enlarged tableau $\widetilde{T}$ as above. For each column C_i of $\widetilde{T}$ define ϵ_i to be the sign of the permutation of $1, 2, ..., n$ represented by that column read from top to bottom. Then define

$$\epsilon_T = \epsilon_1 ... \epsilon_n.$$

Theorem 1.12. *The* N*-th graded component of the isomorphism* $\widetilde{\Psi}$ *is diagonal relative to the standard bases for* $R_{\mathbf{k}}^{(N)}$ *and* $R_{\mathbf{l}}^{(N)}$.

Moreover we have the formula

$$\widetilde{\Psi}(f_T) = \epsilon_T f_{*T}.$$

Remark 1.13. *It is not obvious that the map of graded vector spaces given by the formula in the theorem defines a* ring *homomorphism. Even for the case of rectangular Young diagrams and when the weights* r_i *are all equal, a direct algebraic proof using the Plücker relations is not easy and was given in the thesis of D. Ortland - see the proof of Chapter III,Theorem 1, in* [DO]. *However we know a fortiori that this map of semistandard tableaux is induced by the ring isomorphism* $\widetilde{\Psi}$.

1.4. **Duality for general semisimple complex Lie groups.** Let θ be the Chevalley involution of the Lie group $GL_n(\mathbb{C})$ whence

$$\theta(g) = (g^t)^{-1}.$$

Then θ carries a standard parabolic subgroup P to its opposite P^{opp} and induces a map $\Theta : G/P \to G/P^{opp}$ given by

$$\Theta(gP) = \theta(g)P^{opp}. \tag{2}$$

Next if χ is a character of P then the $\chi^\theta := \chi \circ \theta$ is a character of P^{opp}. Let $\mathcal{L}_\chi$ and $\mathcal{L}_{\chi^\theta}$ be the corresponding homogeneous line bundles. Then we obtain an isomorphism of line bundles $\widehat{\Theta}$ by defining

$$\widehat{\Theta}([g, z]) = [\theta(g), z].$$

We will prove the following lemma in §2, see Lemma 2.12.

Lemma 1.14.

(1) $\Theta = \Psi$.
(2) $\hat{\Theta} = \hat{\Psi}$.

The critical point here is that with this formulation i.e. using Equation 2 extends the duality map to a duality map of weight varieties for all reductive groups. To avoid complications we will restrict ourselves to the cases that either G is semisimple or $G = GL_n(\mathbb{C})$ in what follows.

Remark 1.15. *In fact we get the same map Θ (for general G) using another description. Let $n(w_0) \in N(T)$ be a representative for the longest element w_0 in the Weyl group. We may assume that it is fixed under θ (see §2). Then we define $R : G/P^{opp} \to G/Q$ (where Q is the standard parabolic conjugate to P^{opp}) by $R(gP^{opp}) = gn(w_0)Q$. The reader will verify that R induces the identity on the flag manifold M^{opp}. Thus if we postcompose θ by R we obtain the same map Θ but we have another presentation in terms of coset spaces. We will abuse notation and also use the same symbol Θ for this new presentation. We have then $\Theta : G/P \to G/Q$ with*

$$\Theta(gP) = \theta(g)n(w_0)Q.$$

The reader will verify that with this description the induced map on line bundles carries $\mathcal{L}_{\chi_\lambda}$ to $\mathcal{L}_{\chi_{\lambda^\vee}}$ where $\lambda^\vee$ is the weight contragredient to λ. In what follows we will use whichever description of Θ is convenient.

We now state two theorems and a conjecture, the analogues of the three theorems above for the quotients of flag manifolds of the group $GL_n(\mathbb{C})$.

1.4.1. *Duality of symplectic quotients.* The following theorem is proved in §2.

Theorem 1.16. *Let K be a semisimple compact Lie group with complexification G and T be a maximal torus with $T \subset K$. Choose a Chevalley involution θ of G such that θ carries K into itself and satisfies $\theta(t) = t^{-1}$ for $t \in T$. Let S be a subtorus of G and $Z(S)$ be the centralizer of S whence $\theta(Z(S)) = Z(S)$. Let $M = K/Z(S)$. Let $\mathbf{r}$ be an element of the moment polyhedron for the action of T on $K/Z(S)$. Then the Chevalley involution induces an isomorphism of Kähler manifolds*

$$\overline{\Theta} : M//_{\mathbf{r}}T \to M//_{-\mathbf{r}}T.$$

1.4.2. *Duality of Mumford quotients.* We next state the corresponding general duality result for Mumford quotients. We continue with the notation of the previous theorem. Let H be the complexification of T. There exists a Borel subgroup B of G such that $B \cap \theta(B) = H$. We let B^{opp} denote $\theta(B)$. We let P be a parabolic subgroup of G containing B and let $P^{opp} = \theta(P)$. We will assume G is the simply-connected group thus the character lattice of H is the weight lattice of the Lie algebra $\mathfrak{g}$. To emphasize this point we make the definition

Definition 1.17. *In what follows we say the symplectic manifold M is* integral *will mean λ is in the weight lattice of $\mathfrak{g}$. Here M corresponds to the orbit of λ. We say $\mathbf{r}$ is* integral *if $\mathbf{r}$ is in the weight lattice.*

Let λ be a dominant weight and assume that the corresponding character χ_λ of H extends to a character of P but does not extend to any larger parabolic. Let $\mathcal{L}_{\chi_\lambda}$ be the homogeneous line bundle over the flag manifold $M = G/P$ with isotropy representation χ_λ^{-1}. Then $\mathcal{L}_{\chi_\lambda}$ is a very ample H–bundle. We let $\mathbf{r}$ be another weight and twist the action of H on $\mathcal{L}_{\chi_\lambda}$ by the character of H associated to $\mathbf{r}$ to obtain $\mathcal{L}_{\chi_\lambda}(\mathbf{r})$. We can then form the Mumford quotient $M//_{\mathbf{r}}H$ in the usual way. Similarly we obtain a very ample H–bundle $\mathcal{L}_{\chi_{\lambda\circ\theta}}(-\mathbf{r})$ over the flag manifold $M^{opp} = G/P^{opp} = G/Q$ with isotropy representation $\chi_{\lambda\circ\theta}$. We will denote the corresponding Mumford quotient by $M^{opp}//_{-\mathbf{r}}H$. We have a bundle isomorphism $\hat{\Theta} : \mathcal{L}_{\chi_\lambda}(\mathbf{r}) \to \mathcal{L}_{\chi_{\lambda\circ\theta}}(-\mathbf{r})$ defined as in the case of $GL_n(\mathbb{C})$. We obtain

Theorem 1.18. *The bundle isomorphism $\hat{\Theta}$ induces an isomorphism of Mumford quotients*

$$M//_{\mathbf{r}}H \cong M^{opp}//_{-\mathbf{r}}H.$$

1.4.3. *An explicit formula for duality on the ring level.* There should be an explicit formula for computing the isomorphism $\widetilde{\Theta}$ of homogeneneous coordinate rings associated to the previous isomorphism in terms of the Littelmann path model [Lit] (or any other model) for the irreducible representations given by the graded summands of the two coordinate rings.

Conjecture 1.19. *In the Littelmann path models for the two homogeneous coordinate rings the isomorphism $\widetilde{\Theta}$ is given by reversing Lakshmibai- Seshadri paths and translating the initial points of the reversed paths to the origin.*

1.5. **Self-duality.** We now suppose that the flag manifold M and the level $\mathbf{r}$ have been chosen so that $\overline{\Theta}$ carries $M//_{\mathbf{r}}H$ into itself. We will then say the torus quotient is *self-dual*. We may then ask

Problem 1.20 (Classical Problem)**.** *For which self-dual torus quotients $M//_{\mathbf{r}}H$ is $\overline{\Theta}$ is equal to the identity?*

The quantum version of the previous question is

Problem 1.21 (Quantum Problem)**.** *For which self-dual irreducible representations V_λ does the Chevalley involution act as a scalar on the zero weight space $V_\lambda[0]$.*

Remark 1.22. *If the Chevalley involution is inner then it is clear that it acts on $V_\lambda[0]$, if not then the action on V_λ and $V_\lambda[0]$ is defined only up to a scalar multiple, see* [MTL]*, §4.3.*

The motivation for this problem is explained in [MTL] where it is solved for the groups $SL_n(\mathbb{C})$ and $G_2(\mathbb{C})$. For each irreducible V_λ as above the authors in [MTL] construct an action of the Artin group $B_{\mathfrak{g}}$ associated to $\mathfrak{g}$ (the fundamental group of the quotient by the Weyl group of the space of regular elements in a Cartan subalgebra) on $V_\lambda[0] \otimes \mathbb{C}[[h]]$. This representation is the monodromy representation of the Casimir connection and by a very recent theorem of Toledano Laredo, see [TL2], coincides with the representation constructed by Lusztig, [Lu], Ch. 41,

using the theory of quantum groups, see also [TL1] where the result was proved for the case of $SL_n(\mathbb{C})$. In [MTL] the authors began a study of the irreducibility of these representations. The starting point of this study of irreducibility was the observation that in case V_λ was self-dual these representations commute with the action of the Chevalley involution and hence if the Chevalley involution does not act as a scalar (as is nearly always the case) they are reducible.

We now describe our solution of the classical problem. The solution of the problem for the groups $GL_n(\mathbb{C})$ and G_2 follows from earlier work of the second author and V. Toledano Laredo, [MTL]. In what follows note that $\mathbf{r} = 0$ for all cases except for (1).

Theorem 1.23.

(1) *Suppose $G = GL_n(\mathbb{C})$. Assume that $\mathbf{k}$ and $\mathbf{r}$ satisfy the self-duality conditions $\mathbf{k} = \mathbf{l}$ and $\mathbf{r} = \mathbf{s}$. The self-duality $\overline{\Theta} : F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H \to F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H$ is equal to the identity if and only if the flag manifold is*
   (a) $\mathbb{CP}^1$ *with the symplectic form corresponding to* $a\varpi_1$ *and* $\mathbf{r} = (a/2)\varpi_2$.
   (b) $F_{\mathbf{k}}(\mathbb{C}^n) = F_{1,n-1}(\mathbb{C}^n)$ *with the symplectic form* $a\varpi_1 + a\varpi_{n-1}$ *and* $\mathbf{r} = a\varpi_n$.
   (c) $F_{\mathbf{k}}(\mathbb{C}^n) = Gr_2(\mathbb{C}^4)$ *with the symplectic form* $2a\varpi_2, a \in \mathbb{N}$ *and* $\mathbf{r} = a\varpi_4$.

(2) *Suppose $G = Sp_{2n}(\mathbb{C})$.Then the duality map $\overline{\Theta}$ is equal to the identity if and only if the flag manifold is*
   (a) *The projective space $\mathbb{CP}^{2n-1}$ with the symplectic form corresponding to a multiple of ϖ_1.*
   (b) *The Lagrangian Grassmannian $Gr_2^0(\mathbb{C}^4)$ with the symplectic form corresponding to a multiple of ϖ_2.*

(3) *Suppose that $G = SO_{2n+1}(\mathbb{C})$. Then the duality map $\overline{\Theta}$ is equal to the identity if and only if the flag manifold is*
   (a) *The quadric hypersurface $\mathcal{Q}_{2n-1} \subset \mathbb{CP}^{2n}$ with the symplectic form corresponding to a multiple of ϖ_1.*
   (b) *The Lagrangian Grassmannian $Gr_2^0(\mathbb{C}^5)$ with the symplectic form corresponding to a multiple of ϖ_2.*

(4) *Suppose that $G = SO_{2n}(\mathbb{C})$. Then the duality map $\overline{\Theta}$ is equal to the identity if and only if the flag manifold is*
   (a) *The quadric hypersurface $\mathcal{Q}_{2n-2} \subset \mathbb{CP}^{2n-1}$ with the symplectic form corresponding to a multiple of ϖ_1.*
   (b) *One of the Lagrangian Grassmannians $Gr_2^0(\mathbb{C}^4)^+ \cong \mathbb{CP}^1$ and $Gr_2^0(\mathbb{C}^4)^- \cong \mathbb{CP}^1$ with the Kähler forms corresponding to a constant curvature form.*
   (c) *The Grassmannian of isotropic two-planes $Gr_2^0(\mathbb{C}^6)$.*
   (d) *One of the Lagrangian Grassmannians $Gr_4^0(\mathbb{C}^8)^+$ and $Gr_4^0(\mathbb{C}^8)^-$ with the symplectic forms corresponding to multiples of ϖ_3 and ϖ_4.*
   (e) *The isotropic flag manifold $F_{1,2}^0(\mathbb{C}^4) \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ with the Kähler form corresponding to the sum of any two constant curvature forms.*

Remark 1.24. *We see from the above that there are three infinite families of examples where the duality is trivial, the line-hyperplane pairs, the lines in symplectic vector spaces and the quadrics for the orthogonal groups. It is remarkable that* all *the other examples are obtained from these three infinite families using exceptional isomorphisms. We discuss two examples in detail. First, the example (4)*

(c), the isotropic Grassmannian $Gr_2^0(\mathbb{C}^6)$, is explained by the exceptional isomorphism $D_3 \cong A_3$ which carries $Gr_2^0(\mathbb{C}^6)$ to the member of the infinite family of line-hyperplane pairs given by $F_{1,3}(\mathbb{C}^4)$. Second, the example of the two Lagrangians in (4)(d), is explained by triality. Indeed we have

$$Gr_4^0(\mathbb{C}^8)^+ \cong Gr_4^0(\mathbb{C}^8)^- \cong \mathcal{Q}_6.$$

As for the other two not-quite-obvious examples, (2)(b) and (3)(b), we have, using the exceptional isomorphism $C_2 \cong B_2$,

(1) $Gr_2^0(\mathbb{C}^4) \cong \mathcal{Q}_3$.
(2) $Gr_2^0(\mathbb{C}^5) \cong \mathbb{CP}^3$.

1.6. **The idea of the proof.** We conclude the mathematical part of this Introduction with a sketch of the proof of the previous theorem. The following definition is critical in what follows.

Definition 1.25. *A representation V_λ is* good *if it is self-dual and for some N the Chevalley involution θ does not act as a scalar on $V_{N\lambda}[0]$.*

We prove in what follows that if $M//_0H$ is a self-dual torus quotient then the duality map $\overline{\Theta}$ is nontrivial if and only if the representation V_λ is good, Theorem 6.7. Here we assume that M is the flag manifold associated to λ. We also prove that the good representations are closed under Cartan products, see Definition 6.1. In fact any Cartan product of self-dual representations is good provided at least one factor is good, Theorem 6.4. Thus,for example, to prove that all self-dualities are nontrivial for a given group G such that $-1 \in W$ we have only to prove that all the fundamental representations are good. We prove this for the groups G_2, F_4, E_7 and E_8 by branching a fundamental representation to a carefully chosen maximal subgroup of maximal rank and observing that this restriction contains either a good representation or a nonself-dual representation.

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2. The Chevalley involution and duality of symplectic quotients

We recall the definition of a Chevalley involution. Choose a Cartan subalgebra $\mathfrak{h}$ and a Borel $\mathfrak{b}$ containing $\mathfrak{h}$. Thus we obtain a system of roots R together with a positive subsystem $R_+ \subset R$ and a simple subsystem $S \subset R_+$. For each simple root α choose a root vector x_α corresponding to α. Let $h_\alpha \in \mathfrak{h}$ be the coroot corresponding to α. Then there is a unique negative root vector $x_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. We then have the following consequence of the Chevalley presentation of $\mathfrak{g}$.

Lemma 2.1. *There exists a unique involutive automorphism θ of $\mathfrak{g}$ such that*

(1) $\theta(x_\alpha) = -x_{-\alpha}$ *for all* $\alpha \in S$.

(2) $\theta(x_{-\alpha}) = -x_\alpha$ *for all* $\alpha \in S$.
(3) $\theta(h_\alpha) = -h_\alpha$ *for all* $\alpha \in S$.

We will say a holomorphic involution of a simple complex Lie algebra $\mathfrak{g}$ is a Chevalley involution if θ satisfies the above formulas for some $\mathfrak{h}$, $\mathfrak{b}$ and vectors x_α,$x_{-\alpha}$ and h_α, $\alpha \in S$.

Remark 2.2. *Any two Chevalley involutions θ_1 and θ_2 are conjugate by Proposition 2.8 of* [ABV]. *Furthermore it is possible to choose root vectors x_α for all positive roots α such that* (1) *and* (2) *continue to hold. This follows provided one has chosen the structure constants $N_{\alpha,\beta}$ for $\mathfrak{g}$ in the Chevalley basis so that*

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}.$$

See [Sam]*, Ch II, §9.*

In our study of self-duality for the weight varieties associated to the exceptional groups we will need the following lemma.

Lemma 2.3. *Suppose H is a Cartan subgroup of a simple complex Lie group G. Suppose θ_1 and θ_2 are holomorphic involutions of G which carry H into itself and satisfy $\theta_i(h) = h^{-1}, h \in H, i = 1, 2$. Then there exists $h \in H$ such that*

$$\theta_2 = Adh \circ \theta_1 \circ Adh^{-1}.$$

Moreover both θ_1 and θ_2 are Chevalley involutions.

Proof. Let α be a root and $\mathfrak{g}_\alpha$ be the corresponding root space. Then we have

$$\theta_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}, i = 1, 2.$$

Let $\alpha_i, 1 \le i \le l$ be the simple roots. For each simple root α_i choose a Chevalley basis vector x_{α_i}. Since $\theta_1(x_{\alpha_i})$ and $\theta_2(x_{\alpha_i})$ both lie in the one dimensional space $\mathfrak{g}_{-\alpha}$ there exist complex numbers c_i such that

$$\theta_2(x_{\alpha_i}) = c_i\theta_1(x_{\alpha_i}).$$

Choose $h \in H$ such that $Adh(x_{\alpha_i}) = \sqrt{c_i}x_{\alpha_i}$ whence $Adh(x_{-\alpha_i}) = (1/\sqrt{c_i})x_{-\alpha_i}$. Then $\theta_2 = Adh^{-1} \circ \theta_1 \circ Adh$. □

In what follows we will have a distinguished Cartan H. The above lemma allows us to make an abuse of language and refer to *the* Chevalley involution of G (and H).

2.0.1. *The action of the Chevalley involution on a self-dual representation.* In this subsection we recall how θ acts on the weight space $V_\lambda[0]$ of a self-dual representation V_λ, see [MTL], §4.3. Indeed because V_λ is self-dual there exists $\Theta_{V_\lambda} \in Aut(V_\lambda)$ of order 2 which intertwines the action ρ of $GL_n(\mathbb{C})$ with the action $\rho^\theta = \rho \circ \theta$ on the same space. By Schur's Lemma Θ_{V_λ} is unique up to multiplication by ± 1. Then the action of θ on V_λ is defined to be the action of Θ_{V_λ}. It is then immediate that Θ_{V_λ} carries the zero weight space $V_\lambda[0]$ into itself.

Lemma 2.4. *Suppose H is a Cartan subgroup of a simple complex Lie group G. Suppose θ_1 and θ_2 are holomorphic involutions of G which carry H into itself and satisfy $\theta_i(h) = h^{-1}, h \in H, i = 1, 2$ Let V be a self-dual irreducible representation of G and let $\Theta_V^{(1)}$ and $\Theta_V^{(2)}$ be the operators assigned to θ_1 and θ_2 according to the rule explained in the preceding paragraph. Then $\Theta_V^{(1)}$ and $\Theta_V^{(2)}$ are conjugate by an*

element of H acting on V and consequently the restrictions of $\Theta_V^{(1)}$ and $\Theta_V^{(2)}$ to $V[0]$ coincide.

2.0.2. *The duality map.* Let P be a standard parabolic subgroup of G and M be the flag manifold G/P. Let $P^{opp} = \theta(P)$ and Q be the standard parabolic subgroup conjugated to P^{opp}. Let $M^{opp} = G/P^{opp}$ and $N = G/Q$. Then $M^{opp} = N$. We have defined the map $\Theta : M \to M^{opp}$

$$\Theta(gP) = \theta(g)P^{opp}.$$

Equivalently we have defined the duality map $\Theta : M \to N$ by

$$\Theta(gP) = \theta(g)n(w_0)Q.$$

It is immediate that

$$\Theta(gx) = \theta(g)\Theta(x), x \in M \tag{3}$$

Let $v \in \mathfrak{g}$ and V_M be the fundamental vector field associated to v. Then the infinitesimal version of Equation (3) follows immediately from Equation (3). We will need it below so we state it as a lemma.

Lemma 2.5. *$\Theta_*(V_M)$ is the fundamental vector field $\theta(v)_N$ on N associated to $\theta(v) \in \mathfrak{g}$.*

As a consequence of Equation (3) and Lemma 2.5 we have

Lemma 2.6.

(1) $\Theta(hx) = h^{-1}\Theta(x), h \in H, x \in M.$
(2) $\Theta_*(V_M) = -V_N, v \in \mathfrak{h}.$

We now prove

Lemma 2.7. *Θ is a Kähler isomorphism.*

Proof. Since $\theta : G \to G$ is holomorphic, any map of quotients it induces is also holomorphic (by the universal property of quotients). Also any automorphism of a Lie algebra induces an isometry of the Killing form (see [Sam], pg. 14). Since in the semisimple case the metric on M is induced by the negative of the Killing form on $\mathfrak{k}$ we are done in the semisimple case. For the case of $GL_n(\mathbb{C})$ we replace the Killing form by the trace form and argue analogously. $\square$

2.1. **The action of the Chevalley involution on the momentum map.** The following result will tell us how Θ relates momentum levels for symplectic quotients. Recall that $N = M^{opp}$ so we have $\Theta : M \to N$.

Proposition 2.8. *Let μ_M and μ_N be the momentum maps for the actions of T on M and N respectively. Then there exists an element $\mathbf{\Lambda} \in (\mathfrak{t}^*)^W$ such that*

$$\Theta^*\mu_N = \mathbf{\Lambda} - \mu_M.$$

Proof. For $v \in \mathfrak{t}$ we let V_M and V_N be the associated fundamental vector fields on M and N respectively. Let h_v^M and h_v^N be the Hamiltonian potentials of V_M and V_N. By Lemma 2.6 we have

$$\Theta_*(V_M) = -V_N.$$

We claim that there exists a linear functional $\mathbf{\Lambda} \in \mathfrak{t}^*$ such

$$\theta^* h_v^N = \mathbf{\Lambda}(v) - h_v^M.$$

To prove the claim it suffices to prove the differentiated version

$$\Theta^* dh_v^N = -dh_v^M.$$

Let $p \in M$ and $w \in T_p(M)$. Then we have

$$\begin{aligned}\Theta^* dh_v^N|_p(w) &= \Theta^*(\iota_{V_N(\Theta(p))}\omega_N|_{\Theta(p)}(d\Theta|_p(w))) = \omega_N|_{\Theta(p)}(V_N(\Theta(p)), d\Theta|_p(w)) \\ &= -\omega_N|_{\Theta(p)}(d\Theta|_p(V_N(p)), d\Theta|_p(w)) = -\omega_M(V_M(p), w) = -\iota_{V_M(p)}\omega_M(w).\end{aligned}$$

The claim follows.

It remains to prove that $\mathbf{\Lambda}$ is invariant under the Weyl group. We first establish the W–equivariance of $\Theta^*\mu_N$. To this end let $x \in M$ and $w \in W$ and let $n(w) \in N(T)$ be a representative of w in $N(T)$, the normalizer of W in $U(n)$. We claim that we may choose these representatives such that $\theta(n(w)) = n(w)$. Indeed the Tits representatives, see [MTL], §2.5, $\exp(x_\alpha)\exp(-x_{-\alpha})\exp(x_\alpha)$ have this property because $\exp(x_\alpha)\exp(-x_{-\alpha})\exp(x_\alpha) = \exp(-x_{-\alpha})\exp(x_\alpha)\exp(-x_{-\alpha})$ as can be checked by a computation in $\mathfrak{sl}_2(\mathbb{C})$. We next observe that it is an immediate consequence of the K–equivariance of the momentum map for the action of K on M (and the relation between the K and T momentum maps) that

$$\mu_M(n(w)\cdot x) = Ad^* w(\mu_M(x)).$$

Then we have

$$\Theta^*\mu_N(n(w)\cdot x) = \mu_N(\Theta(n(w)\cdot x)) = \mu_N(n(w)\Theta(x)) = Ad^* w\mu_N(\Theta(x)) = Ad^* w(\Theta^*\mu_N)(x).$$

Since μ_M is also W–equivariant we find that $\mathbf{\Lambda} = \Theta^*\mu_N + \mu_M$ is also W–equivariant. Thus, since $\mathbf{\Lambda}(x)$ is a constant function $\mathbf{\Lambda} = \mathbf{\Lambda}(wx) = Ad^* w(\mathbf{\Lambda}(x)) = Ad^* w(\mathbf{\Lambda})$. ☐

Corollary 2.9. *If G is semisimple then $\mathbf{\Lambda} = 0$ and we have*

$$\Theta^*\mu_N = -\mu_M.$$

We obtain a general isomorphism formula for the action of Θ on symplectic quotients.

Theorem 2.10. *The map $\Theta : M \to N$ induces a homeomorphism (Kähler isomorphism in the smooth case)*

$$\Theta : M//_{\mathbf{r}}T \to N//_{\mathbf{\Lambda}-\mathbf{r}}T.$$

2.2. **Formulas for the Chevalley involution.** The above isomorphisms will be more useful if we have more explicit formulas for Θ. We begin with a remarkably useful lemma. It will apply to all simple complex groups except $SL_n(\mathbb{C})$ and E_6 (the case of $SO_{4n+2}(\mathbb{C})$ will require a slight modification, see below).

Lemma 2.11. *Suppose the Chevalley involution is inner with $\theta = AdJ$. Let P be a parabolic subgroup and $M = G/P$. Then $\Theta : M \to M$ coincides with the map induced by action of J by left translation.*

Proof. The lemma is obvious if we use the description of Θ from Remark 1.15. If $\theta = AdJ$ then we may take J as a representative of $n(w_0)$ and we obtain $\Theta(gP) = AdJ(g)JP = JgP$. ☐

Assume first that G is a classical group other than $SL_n(\mathbb{C})$ or $SO_{4n+2}(\mathbb{C})$. In what follows J (or J_G) will denote an element of G such that $\theta(g) = AdJ(g)$. Let Q_n denote the n by n matrix with 1's on the counter-diagonal and 0's elsewhere. We have

$$J_{Sp_{2n}(\mathbb{C})} = \begin{pmatrix} 0 & Q_n \\ -Q_n & 0 \end{pmatrix}, \quad J_{SO_{4n}(\mathbb{C})} = Q_{4n}, \quad J_{SO_{2n+1}(\mathbb{C})} = (-1)^n Q_{2n+1}.$$

In the case of $SO_{4n+2}(\mathbb{C})$ there is an element again denoted J in $O_{4n+2}(\mathbb{C})$ such that $\theta(g) = AdJ(g)$. Here J is simply Q_{4n+2}.

Note that in all cases J is a scalar multiple of the matrix of the bilinear form relative to the standard basis.

2.3. **Computations for $GL_n(\mathbb{C})$.**

We begin by relating the duality maps Ψ and $\hat{\Psi}$ of the Introduction to the maps Θ and $\hat{\Theta}$ as promised in the Introduction.

Lemma 2.12.

(1) $\Theta = \Psi$.
(2) $\hat{\Theta} = \hat{\Psi}$.

Proof. We first give the proofs for Grassmannians. To prove (i) first note that the j-th column $C_j(\theta(g))$ is the j–th row of g^{-1} whence

$$(C_j(\theta(g)), C_i(g)) = \delta_{ij}.$$

Here $(\ ,\ )$ denotes the form B. Hence the last $l = n - k$ columns of $\theta(g))$ are orthogonal to the first k columns of g and we have a commutative diagram

$$\begin{array}{ccc} G/P & \xrightarrow{\Theta} & G/P^{opp} \\ {\scriptstyle \pi_k}\downarrow & & \downarrow{\scriptstyle \pi_l} \\ Gr_k(\mathbb{C}^n) & \xrightarrow{\Psi} & Gr_l(\mathbb{C}^n) \end{array}$$

Here the vertical arrows are given by $\pi_k(gP) = g \cdot e_1 \wedge \cdots \wedge e_k$ and $\pi_l(gP^{opp}) = g \cdot e_{k+1} \wedge \cdots \wedge e_n$. the first statement follows.

To prove the second statement we have only to observe there is another commutative square.

$$\begin{array}{ccc} G \times_P \mathbb{C} & \xrightarrow{\hat{\Theta}} & G \times_{P^{opp}} \mathbb{C} \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_l} \\ \mathcal{T}_k & \xrightarrow{\hat{\Psi}} & \mathcal{T}_l \end{array}$$

Here the vertical arrows are given by $f_k([g,z]) = zg \cdot e_1 \wedge \cdots \wedge e_k$ and $f_l([g,w]) = wg \cdot e_{k+1} \wedge \cdots \wedge e_n$. The bundle on the upper left is obtained from the equivalence relation $(g,z) \sim (gp, det_k^{-1}(p)z)$ and the bundle on the upper right is obtained from the equivalence relation $(g,w) \sim (gp, det_l^{-1}(p)w)$.

The reader will verify that the statements in the lemma may now be deduced for the case of flag manifolds for $GL_n(C)$ by comparing the diagram

$$\begin{array}{ccc} F_{\mathbf{k}}(\mathbb{C}^n) & \xrightarrow{\Psi} & F_{\mathbf{l}}(\mathbb{C}^n) \\ i\Big\downarrow & & \Big\downarrow i \\ \prod_{i\leq m} Gr_{k_i}(\mathbb{C}^n) & \xrightarrow{F\circ\prod_i \Psi_i} & \prod_{i\leq m} Gr_{l_i}(\mathbb{C}^n) \end{array}$$

with the analogous diagram where Ψ is replaced by Θ and the factors Ψ_i are replaced by the factors Θ_i.

□

In order to apply Proposition 2.10 we need to compute $\mathbf{\Lambda}$. We have seen that $\mathbf{\Lambda} = 0$ in the semisimple case. We now compute $\mathbf{\Lambda}$ for $GL_n(\mathbb{C})$.

Proposition 2.13.
Suppose that μ_M takes values in the orbit corresponding to the n–tuple of eigenvalues (arranged in weakly decreasing order) $\lambda = (\lambda_1, \ldots, \lambda_n)$ and μ_N takes values in the orbit corresponding to the n–tuple of eigenvalues $(\nu_1, \ldots, \nu_n)$. Then

$$\mathbf{\Lambda} = (\lambda_1 + \nu_n)\varpi_n.$$

We will apply the Proposition to the case where λ_n and μ_n are both zero. We record the resulting formula in the form we will use it.

Corollary 2.14. *Suppose that μ_M takes values in the orbit of $\sum_{i=1}^{n-1} a_i\varpi_i$ and μ_N takes values in the orbit of $\sum_{i=1}^{n-1} b_i\varpi_i$. Then*

$$\mathbf{\Lambda} = (\sum_{i=1}^{n-1} a_i)\varpi_n.$$

Proof. With the assumptions of the corollary we have $\lambda_1 = \sum_{i=1}^{n-1} a_i$ and $\nu_n = 0$. □

We need a preliminary lemma. Let $\mathbf{e}$ be the standard coordinate flag in $\mathbb{C}^n$ $\mathbf{e} = (\mathbb{C}e_1, \mathbb{C}e_1 \wedge e_2, \ldots, \mathbb{C}e_1 \wedge \cdots \wedge e_{n-1})$.

Lemma 2.15. *Suppose M is the manifold of full flags in $\mathbb{C}^n$. Then we have*

(1) $\Theta(\mathbf{e}) = (\mathbb{C}e_n, \mathbb{C}e_{n-1} \wedge e_n, \ldots, \mathbb{C}e_2 \wedge \cdots \wedge e_n)$.
(2) *Suppose μ_M is normalized so that it takes values in the orbit with smallest eigenvalue equal to zero. Then*

$$\mu_M(\mathbf{e}) = \sum_{i=1}^{n-1} a_i\varpi_i.$$

Proof. The first statement follows from the first statement in Lemma 2.12.

The second statement follows from an explicit computation of the moment map for the Grassmannian $Gr_k(\mathbb{C}^n)$ with the symplectic structure given by embedding it as the orbit of ϖ_k. The moment map is given in [GGMS] as follows. For an n by k matrix A, define $\mu_i([A])$, for $1 \leq i \leq n$, as

$$\mu_i([A]) = \frac{\sum_{i\in J} |\det A(J)|^2}{\sum_J |\det A(J)|^2},$$

where J ranges over the k-element subsets of $\{1, \ldots, n\}$, and $A(J)$ is the k by k submatrix of A whose rows are the rows of A indexed by J. Then $\mu = (\mu_1, \ldots, \mu_n)$ is the moment map for the torus. We see then that the value of the moment map

at the standard coordinate k–plane is ϖ_k. If we change the symplectic structure to be the one corresponding to the orbit of $a\varpi_k$ then the value at $\mathbf{e}$ will be $a\varpi_k$. Since the moment map for the full flag manifold under the diagonal torus action is the sum of the moment maps for each factor the second statement follows. □

Remark 2.16. *The point here is to check that the value ϖ_k is attained at the flag* $\mathbf{e}$.

The Proposition is now a consequence of the following lemma

Lemma 2.17.
We have the following identity of $\mathfrak{t}^*$–valued functions on M:

$$\mu_M + \mu_N \circ \Theta = (\lambda_{max} \circ \mu_M + \lambda_{min} \circ \mu_N)\varpi_n.$$

Proof. For ease of notation we prove only the special case where the flag manifold is the manifold of full flags. Note first that the difference between the left-hand side and the right-hand side is invariant under the action of $\mathbb{R}^2$ that translates μ_M by $s\varpi_n$ and translates μ_N by $t\varpi_n$. Hence it suffices to prove the formula in the case that the $\lambda_{min} \circ \mu_M = 0$ and $\lambda_{min} \circ \mu_N = 0$. Hence we may assume that μ_M and μ_N are as in the corollary. We now compute both sides on the standard coordinate flag $\mathbf{e}$. This will determine $\mathbf{\Lambda}$. By the first statement in Lemma 2.15 we have

$$\Theta(\mathbf{e}) = \Psi(\mathbf{e}) = (\mathbb{C}e_n, \mathbb{C}e_{n-1} \wedge e_n, \dots, \mathbb{C}e_2 \wedge \cdots \wedge e_n).$$

Hence $\Theta(\mathbf{e}) = n(w_0)(\mathbf{e})$ where $n(w_0)$ is a representative in $N(T)$ for the longest element w_0 in the Weyl group. Hence we have

$$\mu_N(\Theta(\mathbf{e})) = \mu_N(n(w_0)(\mathbf{e})) = Ad^* w_0(\mu_N)(\mathbf{e}) = Ad^* w_0(\sum_{i=1}^{n-1} b_i\varpi_i).$$

Note that the last equality followed from the second statement in Lemma 2.15. Thus $\mu_N(\Theta(\mathbf{e}))$ has *first* coordinate equal to zero. On the other hand, by the second statement in Lemma 2.15 we see that $\mu_M(\mathbf{e})$ has first coordinate equal to $\sum_{i=1}^{n-1} a_i$. Since the sum of these two vectors has all components equal we conclude that all components of the sum are equal to $\sum_{i=j}^{n-1} a_i$ whence $\mathbf{\Lambda} = (\sum_{i=1}^{n-1} a_i)\varpi_n$. But it is immediate that the eigenvalues of μ_M are the sums $\sum_{i=j}^{n-1} a_i$. Hence the largest eigenvalue of μ_M is $\sum_{i=1}^{n-1} a_i$ and since the smallest eigenvalue of μ_N is zero by definition the lemma is proved. □

3. The Mumford Quotient

3.1. **Definition of Mumford Quotient.** We refer the reader to [Do] for additional details. Suppose that G is a reductive algebraic group, V is a projective variety, and $\eta : G \times V \to V$ is regular action of G. Let $\pi : \mathcal{L} \to V$ be an ample line bundle over V. A G–linearization of $\mathcal{L}$ is a regular action $\widetilde{\eta} : G \times \mathcal{L} \to \mathcal{L}$ which is linear on fibers and makes the following diagram commute:

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\widetilde{\eta}} & \mathcal{L} \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times V & \xrightarrow{\eta} & V \end{array}$$

Given such a linearization, we automatically get linearizations on all tensor powers $\mathcal{L}^{\otimes N}$ of $\mathcal{L}$. Thus G has an action on sections s of $\mathcal{L}^{\otimes N}$ given by $(g \cdot s)(x) =$

$g \cdot s(g^{-1} \cdot x) = \widetilde{\eta}(g, s(\eta(g^{-1}, x)))$. Let $\Gamma(V, \mathcal{L}^{\otimes N})^G$ denote the G–invariant holomorphic sections of $\mathcal{L}^{\otimes N}$. We define the semistable points of V for the chosen linearization $\widetilde{\eta}$ to be

$$V^{ss}_{\widetilde{\eta}} = \{x \in V \mid (\exists N)(\exists s \in \Gamma(V, \mathcal{L}^{\otimes N})^G)(s(x) \neq 0)\}.$$

The Mumford quotient $V//_{\widetilde{\eta}}\, G$ is defined as the quotient space of $V^{ss}_{\widetilde{\eta}}$ such that two points $x, y \in V^{ss}_{\widetilde{\eta}}$ are identified iff their G–orbit closures (computed in $V^{ss}_{\widetilde{\eta}}$) $cl(G \cdot x)$ and $cl(G \cdot y)$ intersect nontrivially. The Mumford quotient $V//_{\widetilde{\eta}}\, G$ is then a projective variety corresponding to the geometric points of $\mathrm{Proj}(S^G)$ where S^G is the graded ring $\bigoplus_{N \geq 0} \Gamma(V, \mathcal{L}^{\otimes N})^G$.

In the case where we have an action η of the complex torus $H \subset G = GL_n(\mathbb{C})$, and a homogeneous line bundle $\mathcal{L} = G \times_P \mathbb{C}$ over G/P, the set of possible linearizations correspond to the complex characters of H, which are all of the form $(z_1, ..., z_n) \mapsto \prod_{i=1}^n z_i^{r_i}$ where $\mathbf{r} = (r_1, ..., r_n) \in \mathbb{Z}^n$. (See [Do], Chapter 7.) We denote the linearization associated to $\mathbf{r}$ as $V//_{\mathbf{r}}\, H$.

3.2. **Gel'fand-MacPherson duality of Mumford quotients.** In the previous section the Mumford (or categorical) quotient was given for a group acting on a projective variety, but there is an easier definition when the variety V is affine. We may take the trivial line bundle $\mathcal{L}$ where the sections are simply $\mathcal{O}(V)$. Any character χ of G determines a linearization of the trivial line bundle. Semistability is defined as before.

Let $k = m + 1$ and let $\mathcal{L}$ be the trivial line bundle $\mathbb{C}^{n\times k} \times \mathbb{C} \to \mathbb{C}^{n\times k}$. The group $GL_k(\mathbb{C})$ acts on the right of $\mathbb{C}^{n\times k}$ by matrix multiplication. The group H of nonsingular diagonal n by n complex matrices acts on the left of $\mathbb{C}^{n\times k}$.

Let the character $det^a : GL_k(\mathbb{C}) \to \mathbb{C}^*$ and the character $\chi_{\mathbf{r}} : H \to \mathbb{C}^*$ be given. The one–dimensional subgroup $K = \{(zI_n, z^{-1}I_k) : z \in \mathbb{C}^*\}$ of $H \times GL_k(\mathbb{C})$ acts trivially on $\mathbb{C}^{n\times k}$. Let G be the quotient of $H \times GL_k(\mathbb{C})$ by K. The character $\chi_{\mathbf{r}} \times det^a$ defines a character of G iff $|\mathbf{r}| = ak$, and we assume that is the case so that we have a G–linearization of the trivial line bundle.

The quotient by $GL_k(\mathbb{C})$ alone is the Grassmannian $Gr_k(\mathbb{C}^n)$. There is a canonical line bundle $\mathcal{L}_1$ over $Gr_k(\mathbb{C}^n)$ which is the quotient of the pullback of $\mathcal{L}$ over the semistable points. The character $\chi_{\mathbf{r}}$ now defines an H–linearization of $\mathcal{L}_1$. If we now take a Mumford quotient of $Gr_k(\mathbb{C}^n)$ by H using the character $\mathbf{r}$ this results in the quotient $\mathbb{C}^{n\times k}//_{\mathbf{r}\times det^a} H \times GL_k(\mathbb{C})$.

On the other hand, if each $r_i > 0$ then the quotient by H alone is $(\mathbb{C}P^{k-1})^n$. Again there is a canonical quotient bundle $\mathcal{L}_2$ of the trivial bundle pulled back over the semistable points, and the character det^a determines a $GL_k(\mathbb{C})$–linearization of $\mathcal{L}_2$. If we now take the Mumford quotient by $GL_k(\mathbb{C})$ (projective equivalence) using the character det^a we again must get $\mathbb{C}^{n\times k}//_{\mathbf{r}\times det^a} H \times GL_k(\mathbb{C})$.

This is illustrated by the following diagram:

$$\begin{tikzcd}
\mathbb{C}^{n\times(m+1)} \arrow[r, "//_{det^a} GL_{m+1}(\mathbb{C})"] \arrow[d, "//_{\mathbf{r}} H"'] & Gr_{m+1}(\mathbb{C}^n) \arrow[d, "//_{\mathbf{r}} H"] \\
(\mathbb{C}P^m)^n \arrow[r, "//_{det^a} GL_{m+1}(\mathbb{C})"] & \mathcal{M}_{\mathbf{r}}(\mathbb{CP}^m)
\end{tikzcd}$$

where $\mathcal{M}_{\mathbf{r}}(\mathbb{CP}^m)$ denotes the Mumford quotient $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}H$.

4. Duality for torus quotients of Grassmannians on the quantum level

4.1. **The Hodge star operator.** To promote the duality map Ψ to a map of bundles we recall the definition of the complex Hodge star operator $*$. Choose an orientation for $\mathbb{C}^n$ (whence an orientation for $(\mathbb{C}^n)^*$). Let $vol \in \bigwedge^n(\mathbb{C}^n)^*$ be the positively oriented element of unit length for the form induced by B. We will say vol is a complex volume form. The complex volume form induces a map (by interior multiplication) $\alpha : \bigwedge^k(\mathbb{C}^n) \to \bigwedge^{n-k}(\mathbb{C}^n)^*$. The form B induces a map $\beta : \bigwedge^{n-k}(\mathbb{C}^n)^* \to \bigwedge^{n-k}(\mathbb{C}^n)$. We define the complex Hodge star $*$ to be the composition $\beta \circ \alpha$. We note that $*$ is an invertible linear map from $\bigwedge^k(\mathbb{C}^n)$ to $\bigwedge^{n-k}(\mathbb{C}^n)$. The pair Ψ and $*$ induce a map from the trivial $\bigwedge^k(\mathbb{C}^n)$–bundle over $Gr_k(\mathbb{C}^n)$ to the trivial $\bigwedge^{n-k}(\mathbb{C}^n)$–bundle over $Gr_{n-k}(\mathbb{C}^n)$. This bundle map carries the subbundle $\mathcal{T}_k$ to the subbundle $\mathcal{T}_{n-k}$. Consequently it induces a bundle isomorphism from the tautological line bundle $\mathcal{T}_k$ over $Gr_k(\mathbb{C}^n)$ to the tautological line bundle $\mathcal{T}_{n-k}$ over $Gr_{n-k}(\mathbb{C}^n)$ covering Ψ (so technically a bundle isomorphism from $\mathcal{T}_k$ to $\Psi^*\mathcal{T}_{n-k}$). We obtain the following diagram

$$\begin{array}{ccc} \mathcal{T}_k & \longrightarrow & Gr_k(\mathbb{C}^n) \times \bigwedge^k(\mathbb{C}^n) \\ {\scriptstyle \Psi\times *}\big\downarrow & & \big\downarrow{\scriptstyle \Psi\times *} \\ \mathcal{T}_{n-k} & \longrightarrow & Gr_{n-k}(\mathbb{C}^n) \times \bigwedge^{n-k}(\mathbb{C}^n) \end{array}$$

In order to obtain an isomorphism of their duals we dualize the definition of $*$ to obtain a new isomorphism again denoted $*$ from $\bigwedge^k((\mathbb{C}^n)^*)$ to $\bigwedge^{n-k}((\mathbb{C}^n)^*)$. We obtain an induced isomorphism of line bundles homomorphisms $\hat{\Psi}$ covering Ψ from $\mathcal{L}_k$ to $\mathcal{L}_{n-k}$ by dualizing the above diagram.

4.2. **The Homogeneous Coordinate Ring of $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}\, H$.** In this section we describe a basis for the coordinate ring of $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}\, H$. We begin by recalling that the Plücker embedding ι_k of $Gr_k(\mathbb{C}^n)$ is the projective embedding of $Gr_k(\mathbb{C}^n)$ corresponding to the very ample line bundle $\mathcal{L}$. According to the general theory of projective embeddings and line bundles we have an embedding

$$\iota_k : Gr_k(\mathbb{C}^n) \to \mathbb{P}(\Gamma(Gr_k(\mathbb{C}^n), \mathcal{L})^*).$$

It is standard that $\Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}) \cong \bigwedge^k((\mathbb{C}^n)^*)$. In what follows we will need an explicit formula for this isomorphism.

Let $x \in Gr_k(\mathbb{C}^n)$ and $\tau \in \bigwedge^k((\mathbb{C}^n)^*)$. We let $res_x : \bigwedge^k((\mathbb{C}^n)^*) \to \bigwedge^k((x)^*) = \mathcal{L}_x$ be the operation of restriction of covectors to x. If $\tau \in \bigwedge^k((\mathbb{C}^n)^*)$ we let $\widetilde{\tau}$ be the section of $\mathcal{L}$ defined by

$$\widetilde{\tau}(x) = res_x(\tau).$$

The following lemma is then standard

Lemma 4.1. *The map $\tau \to \widetilde{\tau}$ induces an isomorphism $\bigwedge^k((\mathbb{C}^n)^*) \cong \Gamma(Gr_k(\mathbb{C}^n), \mathcal{L})$.*

Let $\theta_i, 1 \le i \le n$ be the basis for $(\mathbb{C}^n)^*$ dual to the standard basis $\epsilon_i, 1 \le i \le n$. For $I = \{i_1, .., i_k\} \subset \{1, 2, ..., n\}$ with $i_1 < i_2 < \cdots i_k$ we define

$$\theta_I = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_k.$$

The linear functions θ_I as I ranges through the k–element subsets of $\{1, 2, ..., n\}$ give a basis for $\bigwedge^k((\mathbb{C}^n)^*)$ and consequently give homogeneous coordinates (the Plücker coordinates) to be denoted $X_{i_1,i_2,...,i_k}$ (or X_I) on the projective space $\mathbb{P}(\bigwedge(\mathbb{C}^n))$,

4.2.1. *A basis for the homogeneous coordinate ring of* $Gr_k(\mathbb{C}^n)$. We begin by noting that we have the following formula for the homogeneous coordinate ring R_k of $Gr_k(\mathbb{C}^n)$ as a graded vector space.

$$R_k = \bigoplus_{N=0}^{\infty} \Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}^{\otimes N}) = \bigoplus_{N=0}^{\infty} V_{N\varpi_k}.$$

Here $V_{N\varpi_k}$ is the irreducible representation of $GL_n(\mathbb{C})$ with highest weight $N\varpi_k$. Let $\widetilde{R}_k = \bigoplus_{N=0}^{\infty} R_k^{(N)}$ be the graded ring $\mathbb{C}[X_{i_1,...,i_k}]$, where $i_1, ..., i_k$ ranges over k-element subsets of $\{1, ..., n\}$. We have seen that $\bigwedge((\mathbb{C}^n)^*) \cong \Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}))$ whence the Plücker coordinates correspond to a basis of the degree one elements of R_k. We obtain a map of rings $\phi : \mathbb{C}[X_{i_1,...,i_k}] \to R_k$. The following lemma is standard.

Lemma 4.2.
(1) *ϕ is onto.*
(2) *The kernel of ϕ is generated by quadratic relations in the Plücker coordinates called the Plücker relations.*

We now define certain elements $f_T \in R_k$ defined by fillings T of rectangular Young diagrams D with k rows by numbers between 0 and n. Let D_N be the rectangular Young diagram with k rows and N columns and let T be a filling of D_N. Let I_i be the entries in the i–th column of T. Define

$$f_I = X_{I_1} X_{I_2} \cdots X_{I_N}.$$

We define $deg(T)$, the degree of T, by

$$deg(T) = N$$

and note that f_T is in the N-th graded summand $R_k^{(N)}$ of R_k. Thus we find $deg(f_T)$, the degree of f_T relative to the above grading, is also N. We can now describe a basis for $R_k^{(N)}$. We recall that a filling T of D_N is said to be semistandard if the entries in each column are strictly increasing and the entries in each row are weakly increasing. We will use $\mathcal{SS}(D, n)$ to denote the set of semistandard fillings of a Young diagram D by the integers $1, 2, ..., n$. We have, see [DO],Chapter I, Theorem 1.

Theorem 4.3. *The functions f_T as $T \in \mathcal{SS}(D_N, n)$ form a basis for $R_k^{(N)}$.*

We will call the basis of R_k given by the set of f_T's with T semistandard, the standard basis of R_k.

For i between 1 and n we define $w_i(T)$ to be the number of times i appears in T and we define the weight $wt(T)$ of T to be the n-tuple $wt(T) = (w_1, \cdots w_n)$. We also define the weight $wt(f_T)$ by $wt(f_T) = wt(T)$. This terminology is justified by the following

Lemma 4.4. *Under the action of H on the graded ring R_k the function f_T is a weight vector of weight $wt(T)$.*

4.2.2. *A basis for the homogeneous coordinate ring S_k of $Gr_k(\mathbb{C}^n)//_{\mathbf{r}}H$.* Let $\mathbf{r} = (r_1, ..., r_n)$ be a tuple of non-negative integers. Let $a = \frac{\sum_{i=1}^n r_i}{k}$ as usual. We assume that a is an integer. We have the following formula analogous to Equation 4.2.1 for the underlying graded vector space of S_k

$$S_k = \bigoplus_{N=0}^{\infty} \Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}^{\otimes Na}(N\mathbf{r}))^H = \bigoplus_{N=0}^{\infty} V_{N\varpi_k}(N\mathbf{r}).$$

Note that if $f_T \in S_k^N$ then we have

$$deg(T) = Na \text{ and } wt(T) = N\mathbf{r}.$$

We now check that this relation between $\mathbf{r}$ and the degree and weight of T is automatically satisfied if we merely assume that $wt(T)$ is an integral multiple of $\mathbf{r}$. We thereby obtain a simpler description of the subring of H–invariants $S_k \subset R_k$.

Lemma 4.5. *S_k is the subring of R_k spanned by the monomials f_T such that $wt(T)$ is a multiple of $\mathbf{r}$.*

Proof. Suppose that $wt(T) = \ell\mathbf{r}$ with ℓ a positive integer. Suppose $deg(T) = M$. Then T is a filling of the k by M rectangle D whence (equating the total number of boxes in D)

$$kM = wt(T) = \ell|\mathbf{r}| \text{ so } deg(T) = M = \ell a.$$

□

As a consequence of the previous lemma, Theorem 4.3 and Lemma 4.4 we obtain a basis for S_k.

Theorem 4.6. *The set of standard basis vectors f_T with weight a multiple of $\mathbf{r}$ is a basis for S_k.*

We will call the resulting basis the standard basis of S_k.

4.3. **The proof of Theorem 1.4.** In this subsection we prove Theorem 1.4. We begin by proving the H–equivariance of $*$ and the bundle map $\hat{\Psi}$. First we deal with $*$.

Lemma 4.7. *Let $u \in \bigwedge^k(\mathbb{C}^n)$ and $g \in GL_n(\mathbb{C})$. Then*

$$*g\ u = det(g)(g^t)^{-1} * u.$$

In particular we have

$$*h\ u = det(h)h^{-1} * u, h \in H.$$

Proof. We recall from the Introduction that $* = \beta \circ \alpha$ where α is given by contraction with the volume form vol and $\beta : \bigwedge^{n-k}((\mathbb{C}^n)^*) \to \bigwedge^{n-k}((\mathbb{C}^n)$ is the map induced by B. We prove suitable equivariance formulae for each of α and β. First we claim that for any $g \in GL_n(\mathbb{C})$ we have

$$\alpha(gu) = det(g)(g\alpha)(u).$$

Indeed let $v \in \bigwedge^{n-k}((\mathbb{C}^n)$

$$\alpha(gu) = vol(gu, v) = vol(gu, gg^{-1}v) = det(g)vol(u, g^{-1}v) = det(g)(g\alpha)(v)$$

We conclude by observing that for $g \in GL_n(\mathbb{C})$ and $\tau \in \bigwedge^{n-k}((\mathbb{C}^n)^*)$ we have

$$\beta(g\tau) = (g^t)^{-1}\beta(\tau).$$

Indeed it suffices to prove that for $v \in \mathbb{C}^n$ we have $b(gv) = (g^t)^{-1}b(v)$ where $b : \mathbb{C}^n \to (\mathbb{C}^n)^*$ is the map induced by the bilinear form B. But this is immediate. $\square$

The next corollary follows by dualizing the previous lemma.

Corollary 4.8. *Let $\eta \in \bigwedge^k((\mathbb{C}^n)^*)$ and $h \in H$. Then we have*

$$*g\ \eta = det(g)^{-1}(g^t)^{-1} * \eta.$$

In particular we have

$$*h\ \eta = det(h)^{-1}h^{-1}\eta.$$

We need another corollary. Let ρ_k be the k–the exterior power of the standard representation of GL_n and define ρ_k^θ by $\rho_k^\theta = \rho_k \circ \theta$. Recall that θ is the Chevalley involution $\theta(g) = (g^t)^{-1}$.

Then we have by the above (since $\widetilde{\Psi_k}$ is equal to the Hodge star on $\bigwedge^k((\mathbb{C}^n)^*)$)

Corollary 4.9.

$$\widetilde{\Psi}_k \circ \rho_k \circ \widetilde{\Psi}_{n-k} = \rho_k^\theta \otimes det^{-1}.$$

Next we deal with $\hat{\Psi} : \mathcal{L}_k^{\otimes ma} \to \mathcal{L}_{n-k}^{\otimes ma}$ and prove the first statement in Theorem 1.4.

Lemma 4.10.

$$\hat{\Psi} \circ h = h^{-1} \circ \hat{\Psi}.$$

Proof. It suffices to prove the lemma for the case $m = 1$. We have

$$h(x, \alpha^{\otimes a}) = (hx, \chi_{\mathbf{r}}(h)(h\alpha)^{\otimes a})$$

whence

$$\hat{\Psi}(h(x, \alpha^{\otimes a})) = (\Psi(hx), \chi_{\mathbf{r}}(h)(*h\alpha)^{\otimes a}) = (h^{-1}\Psi(x), \chi_{\mathbf{r}}(h)(det(h)^{-1}h^{-1} * \alpha)^{\otimes a})$$
$$= (h^{-1}\Psi(x), \chi_{\mathbf{r}}(h)det(h)^{-a}h^{-1}(*\alpha)^{\otimes a}) = (h^{-1}\Psi(x), \chi_{\mathbf{\Lambda}-\mathbf{r}}(h^{-1})h^{-1}(*\alpha)^{\otimes a}) = h^{-1}\hat{\Psi}((x, \alpha^{\otimes a}).$$

$\square$

Now we prove the second statement in Theorem 1.4. Note that the map $\widetilde{\Psi}$ on sections induced by the bundle map $\hat{\Psi}$ is given by

$$\widetilde{\Psi}(s)(x) = \hat{\Psi}(s(\Psi^{-1}(x))).$$

The second statement in Theorem 1.4 follows from

Lemma 4.11. *A section s is H–invariant $\Leftrightarrow$ the section $\widetilde{\Psi}(s)$ is H–invariant.*

Proof. By symmetry it suffices to prove the direction $\Rightarrow$. So assume that s is H–invariant. It suffices to prove that $h^{-1}\widetilde{\Psi} = \widetilde{\Psi}$ for all $h \in H$. But

$$(h^{-1}\widetilde{\Psi})(x) = h^{-1}\widetilde{\Psi}(hx) = h^{-1}\hat{\Psi}(s(\Psi^{-1}(hx))) = h^{-1}\hat{\Psi}(s(h^{-1}(\Psi^{-1}(x))))$$
$$= \hat{\Psi}(h(s(h^{-1}(\Psi^{-1}(x))))) = \hat{\Psi}(s(\Psi^{-1}(x))) = \widetilde{\Psi}(s)(x).$$

$\square$

Later we will need the following immediate consequence of Corollary 4.9.

Lemma 4.12. *Let $\rho_{N,a,k}$ be the representation of $GL_n(\mathbb{C})$ on the vector space of sections $\Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}_k^{\otimes aN})$. Let $\rho_{N,a,k}^\theta = \rho_{N,a,k} \circ \theta$. Then we have*

$$\widetilde{\Psi}_{N,a,k} \circ \rho_{N,a,k} \circ \widetilde{\Psi}_{N,a,n-k} = \rho_{N,a,n-k}^\theta.$$

4.4. **The proof of Theorem 1.12.** In this subsection we prove the formula of the introduction for the action of the isomorphism $\widetilde{\Psi}$ on the standard basis vectors f_T for the homogeneous coordinate ring of $Gr_k(\mathbb{C}^n)$.

The bundle map $\hat{\Psi}$ satisfies the formula

$$\hat{\Psi}(res_x(\tau)) = res_{\Psi(x)}(*(\tau)), \quad \tau \in \bigwedge^k((\mathbb{C}^n)^*)$$

We now have

Lemma 4.13.

$$\hat{\Psi}(\widetilde{\theta_I}) = sgn(I,J)\ \widetilde{\theta_J}.$$

Proof. Let $x \in Gr_k(\mathbb{C}^n)$. Then we have $\widetilde{\theta_I}(x) = res_x(\theta_I)$. Now let $y \in Gr_{n-k}(\mathbb{C}^n)$. We have

$$\widetilde{\Psi}(\widetilde{\theta_I}(y)) = \hat{\Psi}(\widetilde{\theta_I}(\Psi^{-1}y)) = res_{\Psi(\Psi^{-1}(y))}(*\theta_I) = sgn(I,J)\ res_y(\theta_J) = sgn(I,J)\ \widetilde{\theta}(y).$$

□

The following corollary is an immediate consequence of the lemma.

Corollary 4.14.

$$\widetilde{\Psi}(\widetilde{\theta_{I_1}} \otimes \widetilde{\theta_{I_2}} \otimes \cdots \otimes \widetilde{\theta_{I_p}}) = sgn(I_1,J_1)\cdots sgn(I_p,J_p)\ \widetilde{\theta_{J_1}} \otimes \widetilde{\theta_{J_2}} \otimes \cdots \otimes \widetilde{\theta_{J_p}}.$$

Theorem 1.12 is a consequence of the corollary because f_T is the image of a tensor product of the above form under the linear map given by multiplication of sections.

5. Duality for torus quotients of flag manifolds on the quantum level

5.1. **The relation between partial flag manifolds and products of Grassmannians.** In this section we extend our results on duality of homogeneous coordinate rings of torus quotients of Grassmannians to torus quotients of flag manifolds.

5.2. **The homogeneous coordinate ring of the flag manifold.** Let P be a parabolic subgroup of $G = \mathrm{GL}_n(\mathbb{C})$. Then G/P is a partial flag manifold $F_{k_1,\ldots,k_m}(\mathbb{C}^n)$ where $0 < k_1 < k_2 < \cdots < k_m < n$. The ample line bundles over G/P are parametrized by weights $a_1\varpi_{k_1} + \cdots + a_m\varpi_{k_m}$ where each a_i is a positive integer. In what follows, let $\mathbf{k} = (k_1,...,k_m)$ and $\mathbf{a} = (a_1,...,a_m)$ and we will abbreviate the above dominant weight to $\lambda_{\mathbf{a}}$. Choose an m–tuple $\mathbf{a}$ as above. The m-tuple $\mathbf{a}$ corresponds to a character $\chi_{\mathbf{a}}$ of P. We let $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ be the line bundle over $F_{\mathbf{k}}(\mathbb{C}^n)$ with isotropy representation $\chi_{\mathbf{a}}^{-1}$ (so $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ had total space defined by the equivalence relation $(gp, \chi_{\mathbf{a}}(p)z) \sim (g,z)$). We define $V_{\lambda_{\mathbf{a}}} = \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{a}})^*$. The group G acts on $V_{\lambda_{\mathbf{a}}}$ and the flag manifold $F_{k_1,\ldots,k_m}(\mathbb{C}^n)$ is embedded in $\mathbb{P}(V_{\lambda_{\mathbf{a}}})$ as the orbit of the line through a highest weight vector.

In this section we will use the embedding

$$i : F_{\mathbf{k}}(\mathbb{C}^n) \to Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_m}(\mathbb{C}^n)$$

to promote our duality results for Grassmannians to flag manifolds.

The line bundle $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ is very ample and we obtain an equivariant projective embedding

$$\iota : F_{\mathbf{k}}(\mathbb{C}^n) \to \mathbb{P}(\Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\mathbf{a}})^*) = \mathbb{P}((V_{\lambda_{\mathbf{a}}})^*).$$

Accordingly we have the following formula for the homogeneous coordinate ring $R_{\mathbf{k}}$ of $F_{\mathbf{k}}(\mathbb{C}^n)$

$$R_{\mathbf{k}} = \bigoplus_{N=0}^{\infty} \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), (\mathcal{L}_{\mathbf{k}}^{\mathbf{a}})^{\otimes N}) = \bigoplus_{N=0}^{\infty} V_{N\lambda_{\mathbf{a}}}.$$

We have the very ample line bundle $\mathcal{L}_{k_1}^{a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{a_m}$ over the product $Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_p}(\mathbb{C}^n)$.

We will also use $\widetilde{R_{\mathbf{k}}}$ to denote the homogeneous coordinate ring of $Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_p}(\mathbb{C}^n)$. Hence

$$\widetilde{R_{\mathbf{k}}} = \bigoplus_{N=0}^{\infty} \Gamma(Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_p}(\mathbb{C}^n), \mathcal{L}_{k_1}^{\otimes N a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{\otimes N a_m}) = V_{a_1\varpi_1} \otimes \cdots \otimes V_{a_m\varpi_m}.$$

Recall that the irreducible representation $V_{\lambda_{\mathbf{a}}}$ occurs with multiplicity one in the tensor product $V_{a_1\varpi_1} \otimes \cdots \otimes V_{a_m\varpi_m}$. Hence there is a canonical $GL_n(\mathbb{C})$–quotient mapping $\pi : V_{a_1\varpi_1} \otimes \cdots \otimes V_{a_m\varpi_m} \to V_{\lambda_{\mathbf{a}}}$. We let $\alpha = \pi^*$ be the dual map.

The following lemma will be very important in what follows.

Lemma 5.1.

$$i^*(\mathcal{L}_{k_1}^{a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{a_m}) = \mathcal{L}_{\mathbf{k}}^{\mathbf{a}}.$$

Proof. We consider the following diagram

$$\begin{array}{ccc} F_{\mathbf{k}}(\mathbb{C}^n) & \xrightarrow{\Delta} & F_{\mathbf{k}}(\mathbb{C}^n) \times \cdots \times F_{\mathbf{k}}(\mathbb{C}^n) \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \pi} \\ F_{\mathbf{k}}(\mathbb{C}^n) & \xrightarrow{i} & Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_m}(\mathbb{C}^n) \end{array}$$

We need the following simple general observation whose proof we leave to the reader. Let $P \subset Q$ be subgroups of a group G. Consequently we have a projection $\pi : G/P \to G/Q$. Let χ be a character of Q and let $\mathcal{L}$ be the homogeneous line bundle with isotropy representation χ, that is, the line bundle with total space $G \times_Q \mathbb{C}$ where $(g, z) \sim (gq, \chi(q)^{-1}z)$. Then the pull-back of $\mathcal{L}$ to G/P is the homogeneous line bundle with isotropy representation $\chi|P$.

From this observation we find that $\pi^*(\mathcal{L}_{k_1}^{a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{a_m})$ is an outer tensor product of the same form except the isotropy representations are the restrictions of the characters corresponding to the weights $a_i\varpi_{k_i}, 1 \leq i \leq m$ to P. The pull-back of this outer tensor product by the diagonal map Δ gives the inner tensor product. But the inner tensor product of homogeneous line bundles is again homogeneous with isotropy character the product of the characters of the factors. But this product is just the character corresponding to the weight $a_1\varpi_{k_1} + \cdots + a_m\varpi_{k_m}$.

□

Combining these observations we obtain the following commutative diagram.

$$\begin{array}{ccc} F_{\mathbf{k}}(\mathbb{C}^n) & \xrightarrow{\iota} & \mathbb{P}((V_{\lambda_{\mathbf{a}}})^*) \\ i\downarrow & & \downarrow\alpha \\ Gr_{k_1}(\mathbb{C}^n)\times\cdots\times Gr_{k_m}(\mathbb{C}^n) & \longrightarrow & \mathbb{P}((V_{a_1\varpi_1}\otimes\cdots\otimes V_{a_m\varpi_m})^*) \end{array}$$

5.2.1. *The* $\mathbf{r}$*-linearization of the action of* PH. Let H be the complex torus of diagonal matrices in $GL_n(\mathbb{C})$, and let PH be the image of H under the quotient map $GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$. The H–linearizations of $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ correspond to a character $\chi_{\mathbf{r}}$ of H given by $(z_1, ..., z_n) \to \prod_i z_i^{r_i}$, where $\mathbf{r} \in \mathbb{Z}^n$. We denote the linearized bundle associated to this character as $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}(\mathbf{r})$.

Lemma 5.2. *The induced action of* H *on* $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ *corresponding to* $\mathbf{r}$ *descends to the quotient group* PH *iff* $|\mathbf{r}| = \sum_i a_i k_i$.

Proof. Let $h = \mu I$ be a nonzero scalar matrix. Then for $[g, z]$ in the total space of $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}(\mathbf{r})$ we have

$$h[g,z] = [hg, \chi_{\mathbf{r}}(h)z] = [gh, \mu^{|\mathbf{r}|}z] = [g, (\prod_i det_{k_i}^{a_i}(\mu I_n))^{-1}\mu^{|\mathbf{r}|}z] = [g, \mu^{-\sum_i a_i k_i}\mu^{|\mathbf{r}|}z].$$

Thus $h[g, z] = [g, z] \Leftrightarrow |\mathbf{r}| = \sum_i a_i k_i$. □

5.2.2. *The standard basis of* $R_{\mathbf{k}}$. In what follows we will construct the standard basis of $\Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\mathbf{a}})$. Since the basis vectors are weight vectors for H we will also obtain a basis for $\Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\mathbf{a}})(\mathbf{r})$. By replacing $\lambda_{\mathbf{a}}$ by $\lambda_{N\mathbf{a}}$ one obtains the standard basis for the N-th graded summand of the homogeneous coordinate ring $R_{\mathbf{k}}$.

Define a partition $\mathbf{p} = (p_1, ..., p_n)$ by

$$p_i = \sum_{j=i}^{n} a_j.$$

Let D be the Young diagram corresponding to the partition $\mathbf{p}$ (so there are p_i boxes in the i–th row, $1 \le i \le m$). Let T be a filling (not necessarily semistandard but such that the entries in each column are strictly decreasing) of D by elements in $1, 2, ..., n$. Our goal is to construct an element $f_T \in R_{\mathbf{k}}$. The key to doing this is first to construct a section of the line bundle $\mathcal{L}_{k_1}^{a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{a_m}$ over the product $Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_p}(\mathbb{C}^n)$ then pull-back to $F_{\mathbf{k}}(\mathbb{C}^n)$ using Lemma 5.1.

To do this we divide T up into m rectangular subtableaux $T_1, T_2, ..., T_m$ where T_{n+1-i} is a filling of a k_i by a_i rectangle. Thus we take T_1 to be the rectangle that is the union of the last a_1 columns. See Example 5.5.

Next observe that f_{T_i} is a section of the line bundle $\mathcal{L}_{k_i}^{a_i}, 1 \le i \le m$. The tensor product $f_{T_1} \otimes \cdots \otimes f_{T_m}$ is the desired section of $\mathcal{L}_{k_1}^{a_1} \boxtimes \cdots \boxtimes \mathcal{L}_{k_m}^{a_m}$

We define f_T by

$$f_T = i^*(f_{T_1} \otimes \cdots \otimes f_{T_m}).$$

Here i^* denotes the pullback operation from sections of a line bundle to sections of the pull-back bundle.

By Lemma 5.1 we have

Lemma 5.3. f_T *is a section of* $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$.

We now have the following theorem, see [GL], Chapter 7, Theorem 2.1.1.

Theorem 5.4. *The set of sections $\{f_T, T \in \mathcal{SS}(D,n)\}$ is a basis for $\Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\mathbf{a}})$.*

Example 5.5. *We consider the flag manifold $F_{\mathbf{k}}(\mathbb{C}^n) = F_{2,3}(\mathbb{C}^5)$ with the very ample line bundle corresponding to the dominant weight weight $2\varpi_2 + \varpi_3$. The associated Young diagram D is*

$$D=\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\ .$$

Let T be the filling of D given by

$$T=\begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

Hence

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \end{array}.$$

The section f_T is the pull-back to the flag manifold of the section over $Gr_2(\mathbb{C}^5) \times Gr_3(\mathbb{C}^5)$ given by $f_{T_1} \otimes f_{T_2}$, where f_{T_i} is the section of $\mathcal{L}_{k_i}^{a_i}$ over $Gr_{k_i}(\mathbb{C}^n)$.

5.3. **Duality of tableaux.** We define a map $*$ on tableaux as follows. Let T be a tableau. Suppose the i^{th} column of T is $c_i = (p_1, ..., p_\ell)$, with distinct p_j's. Let $d_i = (q_1, ..., q_{n-\ell})$ where $\{p_1, ..., p_\ell, q_1, ..., q_{n-\ell}\} = \{1, ..., n\}$, and $q_t < q_{t+1}$ for all t. Let $*T$ be the tableau whose i^{th} column is d_{n-i+1} for $1 \le i \le n$.

Example 5.6. $T = \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \Longrightarrow \begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & 2 & 1 \\ \hline 1 & 3 & 3 \\ \hline 4 & 5 & 4 \\ \hline \end{array} \Longrightarrow *T = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 3 & 3 & 4 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$

Theorem 5.7. *The map $*$ takes semistandard tableaux to semistandard tableaux.*

Proof. Let $[n]$ denote the set $\{1, ..., n\}$. We define a partial order on the subsets of $[n]$ given by $I \le J$ iff $|I| \ge |J|$ and $i_q \le j_q$ for all $q \le t$, where $I = \{i_1, ..., i_s\}$ and $J = \{j_1, ..., j_t\}$ have elements listed in strictly increasing order. Define $*I$ as the complement of I in $[n]$. We show that $I \le J$ implies $*I \ge *J$ by induction on n. If $n = 1$ this is trivial. Now suppose that the statement is true for $n-1$. Let $I_{n-1} = I \cap [n-1]$ and let $J_{n-1} = J \cap [n-1]$. Define $*I_{n-1} = [n-1] \setminus I_{n-1}$ and $*J_{n-1} = [n-1] \setminus J_{n-1}$. Clearly $I_{n-1} \le J_{n-1}$. By the induction hypothesis, $*I_{n-1} \ge *J_{n-1}$. Suppose there is some $*j_q > *i_q$, where $*j_q$ is the q^{th} element of $*J$ and $*i_q$ is the q^{th} element of $*I$. Since $*J_{n-1} \le *I_{n-1}$, we have that $|*J_{n-1}| \ge |*I_{n-1}|$ and thus $q = 1 + |*I_{n-1}|$ and so $n = *i_q \in *I$, a contradiction with $*j_q > *i_q$.

The columns of $*T$ are stictly increasing by definition. Hence we need only show that the rows are weakly increasing. Let $*t_{i,j}$ denote the (i,j)–entry of $*T$. We must show that $*t_{i,j} \le *t_{i,j+1}$ for all (i,j) in the valid range. Let d_j, d_{j+1} be adjacent columns in $*T$. Then the respective complementary columns c_{n-j+1}, c_{n-j} are adjacent columns of T. Since T is semistandard, the sets I, J of the entries of

c_{n-j}, c_{n-j+1} respectively are such that $I \leq J$ for the partial order on subsets of $[n]$ mentioned above. Hence $*I \geq *J$, and since $*I$ corresponds to d_{j+1} and $*J$ corresponds to d_j, we have that $*t_{i,j} \leq *t_{i,j+1}$ for all i. □

Note that if the columns of T are strictly increasing, then $**T = T$.

We conclude our discussion of duality of tableaux with a formula for how $*$ changes the weights. Let $w_i(T)$ be the number of times the index i appears in T, and let $wt(T) = (w_1(T), ..., w_n(T))$. Let $D_{\mathbf{a}}$ be the Young diagram with m rows so that the i-th row has length $a_1 + a_2 + \cdots + a_{m-i+1}$.

Theorem 5.8. *For all tableaux T with diagram $D_{\mathbf{a}}$*

$$wt(T) + wt(*T) = \mathbf{\Lambda} = (|\mathbf{a}|, |\mathbf{a}|, ..., |\mathbf{a}|).$$

Proof. The j^{th} column of T and the $(n-j+1)^{th}$ column of $*T$ partition the set $\{1, ..., n\}$. The total number of columns in either tableau is $|\mathbf{a}| = \sum_i a_i$. Fix any $i \in \{1, ..., n\}$. Let c_j denote the j^{th} column of T and let d_j denote the $(n-j+1)^{th}$ column of $*T$. The index i is in exactly one of c_j, d_j. Hence $w_i(T) + w_i(*T) = |\mathbf{a}|$, the total number of columns. □

5.4. **Duality of Mumford quotients for flag manifolds.**

5.4.1. *The fundamental diagram.* The orthogonal complement map Ψ maps the flag $F_{k_1,...,k_m}(\mathbb{C}^n)$ to $F_{l_1,..,l_m}(\mathbb{C}^n$ where $l_i = n - k_{p-i+1}$. Denote $\mathbf{l} = (l_1, ..., l_m)$. Let $\mathbf{\Lambda} = (|\mathbf{a}|, |\mathbf{a}|, ..., |\mathbf{a}|)$ and let $\mathbf{b} = (a_m, a_{m-1}, ..., a_2, a_1)$, the tuple $\mathbf{a}$ in reverse order,

The extension of our duality theorem from Grassmannians to flag manifolds will result from considering the diagram

$$\begin{tikzcd}
F_{\mathbf{k}}(\mathbb{C}^n) \arrow[r, "\Psi"] \arrow[d, "i"'] & F_{\mathbf{l}}(\mathbb{C}^n) \arrow[d, "i"] \\
Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_m}(\mathbb{C}^n) \arrow[r, "F \circ \prod_i \Psi_i"] & Gr_{l_1}(\mathbb{C}^n) \times \cdots \times Gr_{l_m}(\mathbb{C}^n)
\end{tikzcd}$$

We will refer to this diagram as the *fundamental diagram* in what follows.

5.4.2. *The bundle isomorphism $\hat{\Psi}$ and the induced isomorphism of rings of sections.*

Lemma 5.9. *There is a bundle isomorphism $\hat{\Psi}$ covering the isomorphism Ψ : $F_{\mathbf{k}}(\mathbb{C}^n) \to F_{\mathbf{l}}(\mathbb{C}^n)$. Moreover $\hat{\Psi}$ satisfies*

$$\hat{\Psi} \circ h = h^{-1} \circ \hat{\Psi}.$$

Proof. We have seen in our analysis of duality for Grassmannians that the isomorphism Ψ_i can be covered by a bundle isomorphism $\hat{\Psi}_i$ satisfying

$$\hat{\Psi} \circ h = h^{-1} \circ \hat{\Psi}.$$

Hence the isomorphism $F \circ \Pi_i \Psi_i$ is covered by the bundle isomorphism $F \circ \Pi_i \hat{\Psi}_i$ which satisfies the above equivariance condition with respect to the product $H \times \cdots \times H$ and hence a fortiori with respect to the diagonal. But by Lemma 5.1 the bundles $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ and $\mathcal{L}_{\mathbf{k}}^{\mathbf{b}}$ are pull-backs by i of the corresponding bundles on the products of Grassmannians. Hence the pull-back of $F \circ \Pi_i \Psi_i$ by i is a bundle isomorphism from $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$ to $\mathcal{L}_{\mathbf{k}}^{\mathbf{a}}$. □

We obtain induced isomorphisms $\widetilde{\Psi} : \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\mathbf{a}\otimes N}) \to \Gamma(F_{\mathbf{l}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{l}}^{\mathbf{b}\otimes N})$ by the formula

$$\widetilde{\Psi}(s)(x) = \hat{\Psi}(s(\Psi^{-1})).$$

Later we will need that $\widetilde{\Psi}(s)(x)$ intertwines the representation $\rho_{\mathbf{k}}$ with the action $\rho_{\mathbf{l}}^{\theta}$ where $\rho_{\mathbf{l}}^{\theta} = \rho_{\mathbf{l}} \circ \theta$. This follows immediately from Lemma 4.12. We state this result as a lemma.

Lemma 5.10. *Let $\rho_{N,a,\mathbf{k}}$ be the representation of $GL_n(\mathbb{C})$ on the vector space of sections $\Gamma(Gr_k(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{\otimes aN})$. Let $\rho_{N,a,\mathbf{k}}^{\theta} = \rho_{N,a,\mathbf{k}} \circ \theta$. Then we have*

$$\widetilde{\Psi}_{N,a,\mathbf{k}} \circ \rho_{N,a,\mathbf{k}} \circ \widetilde{\Psi}_{N,a,\mathbf{l}} = \rho_{N,a,\mathbf{l}}^{\theta}.$$

We next compute the action of the ring isomorphism $\widetilde{\Psi}$ on the elements f_T and thereby determine how it changes weights of H.

Theorem 5.11.

$$\widetilde{\Psi}(f_T) = \epsilon_T f_{*T}.$$

Proof. We have the following diagram of homogeneous coordinate rings corresponding to the fundamental diagram.

$$\begin{array}{ccc} R_{\mathbf{k}} & \xrightarrow{\widetilde{\Psi}} & R_{\mathbf{l}} \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ \widetilde{R_{\mathbf{k}}} & \xrightarrow{F \circ \prod_i \widetilde{\Psi}_i} & \widetilde{R_{\mathbf{l}}} \end{array}$$

Let T be a tableau and $f_T \in \bar{R}_{\mathbf{k}}$. Then there are tableaux $T_1, ..., T_m$ such that $f_T = i^*(f_{T_1} \otimes \cdots \otimes f_{T_m})$, and $\widetilde{\Psi}(f_T) = i^*(\widetilde{\Psi_1}(f_{T_1}) \otimes \cdots \otimes \widetilde{\Psi_m}(f_{T_m})) = (\prod_i \epsilon_{T_i}) i^*(f_{(*T_1)} \otimes \cdots \otimes f_{(*T_m)}) = \epsilon_T f_{*T}$. $\square$

Corollary 5.12. *$\widetilde{\Psi}$ maps the subspace of $R_{\mathbf{k}}^{(N)}$ of H–weight $N\mathbf{r}$ isomorphically to the subspace of $R_{\mathbf{l}}^{(N)}$ of H–weight $N(\mathbf{\Lambda} - \mathbf{r})$.*

We have now proved one of our main theorems.

Theorem 5.13. *The isomorphism $\widetilde{\Psi}$ induces an isomorphism of graded rings*

$$\bigoplus_{N=0}^{\infty} \Gamma(F_{\mathbf{k}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{k}}^{N\mathbf{a}}(N\mathbf{r}))^H \cong \bigoplus_{N=0}^{\infty} \Gamma(F_{\mathbf{l}}(\mathbb{C}^n), \mathcal{L}_{\mathbf{l}}^{N\mathbf{b}}(N\mathbf{s}))^H$$

and consequently an isomorphism of Mumford quotients

$$F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H \cong F_{\mathbf{l}}(\mathbb{C}^n)//_{\mathbf{s}}H.$$

Proof. The theorem follows immediately Corollary 5.12. $\square$

5.5. **Duality of Kähler structures.** We first observe that it follows from the fundamental diagram on the corresponding result for Grassmannians that $\widetilde{\Psi}$: $F_{\mathbf{k}}(\mathbb{C}^n) \to F_{\mathbf{l}}(\mathbb{C}^n)$ is a holomorphic isometry. We now check that the induced map on quotient is also a holomorphic isometry (hence symplectic). We already know it is holomorphic. We have only to check that it is symplectic.

Theorem 5.14. *The map Ψ induces a homeomorphism of the symplectic quotients:*

$$\overline{\Psi} : F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}T \to F_{\mathbf{l}}(\mathbb{C}^n)//_{\mathbf{s}}T.$$

Furthermore, if $\mathbf{r}$ is a regular value of the momentum mapping, then the symplectic quotients are symplectic manifolds and $\overline{\Psi}$ is a symplectomorphism.

Proof. We recall the fundamental diagram.

$$\begin{tikzcd}
F_{\mathbf{k}}(\mathbb{C}^n) \arrow[r, "\Psi"] \arrow[d, "i"'] & F_{\mathbf{l}}(\mathbb{C}^n) \arrow[d, "i"] \\
Gr_{k_1}(\mathbb{C}^n) \times \cdots \times Gr_{k_m}(\mathbb{C}^n) \arrow[r, "F\circ\prod_i \Psi_i"] & Gr_{l_1}(\mathbb{C}^n) \times \cdots \times Gr_{l_m}(\mathbb{C}^n)
\end{tikzcd}$$

The map $\prod_i \Psi_i$ on the bottom is symplectic, and the inclusion maps are symplectic, so the Ψ map on the top is symplectic. The product T^m acts in a Hamiltonian fashion on each of the two products of Grassmannians. Let T^Δ be the diagonal. Then the inclusion map i is equivariant with respect to T^Δ. Let μ^Δ be the momentum mapping for T^Δ. Hence it suffices to prove that $\prod_i \Psi_i$ carries $(\mu^\Delta)^{-1}(\mathbf{r})$ to $(\mu^\Delta)^{-1}(|\mathbf{a}| - \mathbf{r})$. Choose $\mathbf{x} = (x_1, ..., x_m) \in F_{\mathbf{k}}$ such that $\mu^\Delta(\mathbf{x}) = \mathbf{r}$. Assume $\mu_i(x_i) = \mathbf{r}^{(i)} \in \mathbb{R}^n$. We have $\mu^\Delta = \sum \mu_i$ where μ_i is the momentum mapping for the action of T_i on the i^{th} factor of the product. We have seen that Ψ_i takes $\mu_i^{-1}(\mathbf{r}^{(i)})$ to $\mu_i^{-1}(\mathbf{s}^{(i)})$ where $\mathbf{r}^{(i)} + \mathbf{s}^{(i)} = (a_i, ..., a_i)$. Thus $\mu^\Delta((\prod_i \Psi_i)(\mathbf{x})) = (\sum_i a_i)(1, 1, ..., 1) - \mathbf{r}$. □

5.6. **Duality of Gelfand-Tsetlin systems.**

5.6.1. *The duality map as a map of coadjoint orbits.* In this subsection we will describe the duality map Ψ as a map of coadjoint orbits. We identify the dual $\mathfrak{u}^*(n)$ with the space $\mathcal{H}_n$ of n by n Hermitian matrices using the imaginary part of the trace form. We note that ϖ_n is identified to I_n. We will in fact compute with the map Φ which operates of flags by taking the orthogonal complement relative to the positive definite Hermitian form F. We note that if we use σ to denote complex conjugation we have

$$\Phi = \Psi \circ \sigma.$$

The advantage in using Φ is that Φ is $U(n)$–equivariant.

Let $Flag$ denote the disjoint union of all the flag varieties of various lengths. We define a map $\mathcal{E} : \mathcal{H}_n \to Flag$ as follows. Let $A \in \mathcal{H}_n$ be given. Let $\lambda_{i_1}, ..., \lambda_{i_k}$ be the distinct eigenvalues of A *arranged in decreasing order* and let E_j be the eigenspace belonging to λ_{i_j}. Then we define $\mathcal{E}(A)$ to be the flag of partial sums of the E_j whence

$$\mathcal{E}(A) = (E_1, E_1 + E_2, ..., E_1 + E_2 + \cdots + E_{k-1}).$$

It is important to note that $\mathcal{E}$ loses information. A flag manifold equipped with an invariant symplectic form does not determine a unique orbit. If we change the orbit by adding a multiple of I_n we do not change the flag manifold as a symplectic

manifold. Indeed the map from the orbit to the flag manifold assigns the flag attached to increasing partial sums of eigenspaces and the symplectic form depends only on the differences of the eigenvalues. Note that $\mathbb{R}$ acts on $\mathcal{H}_n$ by translating by multiples of I_n. If we choose a cross-section to this action we can lift Φ and Ψ. We will henceforth choose the cross-section $\mathcal{H}_n^0$ of Hermitian matrices with smallest eigenvalue equal to zero.

Let $\Xi : \mathcal{H}_n \to \mathcal{H}_n$ be the map given by

$$\Xi(A) = \lambda_{max}(A)I_n - A.$$

The reader will check that Ξ is a Poisson map and carries $\mathcal{H}_n^0$ into itself. Let $\Sigma : \mathcal{H}_n \to \mathcal{H}_n$ be complex conjugation so $\Sigma(A) = \overline{A}$.

Lemma 5.15. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_n^0 & \xrightarrow{\Xi} & \mathcal{H}_n^0 \\ \mathcal{E}\big\downarrow & & \big\downarrow\mathcal{E} \\ Flag & \xrightarrow{\Phi} & Flag \end{array}$$

Proof. Suppose that $\lambda_{i_1}, ..., \lambda_{i_k}$ are the distinct eigenvalues of A arranged in decreasing order. Then the eigenvalues of $-A$ arranged in decreasing order are $-\lambda_{i_k}, ..., -\lambda_{i_1}$ and consequently $\mathcal{E}(-A) = \Phi(\mathcal{E}(A))$. □

Corollary 5.16. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_n^0 & \xrightarrow{\Xi\circ\Sigma} & \mathcal{H}_n^0 \\ \mathcal{E}\big\downarrow & & \big\downarrow\mathcal{E} \\ Flag & \xrightarrow{\Psi} & Flag \end{array}$$

Remark 5.17. *We note that (because $A^* = A$)*

$$\Xi \circ \Sigma(A) = \lambda_{max}(A)I_n - A^t = \lambda_{max}(A)I_n + \theta(A).$$

Thus the duality map Ψ lifted to the space of normalized coadjoint orbits is once again given by the Chevalley involution (up to a translation).

5.6.2. *Duality of Gelfand-Tsetlin systems.* We begin by recalling the definition of the Gelfand-Tsetlin Hamiltonians $\lambda_{i,j} : \mathcal{H}_n \to \mathbb{R}, 1 \le i, j \le n$. Let $B_j(A), 1 \le j \le n$, be the upper principal j by j block of A. Then $\lambda_{i,j}(A)$ is the i–th eigenvalue of $B_j(A)$ (the eigenvalues are arranged in (weakly) decreasing order). The $\lambda_{i,j}$'s Poisson commute [GS] and the $\lambda_{i,n}$ are Casimirs. The set of $\lambda_{i,j}$'s is called the Gelfand-Tsetlin system. By restricting the Gelfand-Tsetlin system to any orbit we obtain an integrable system on that orbit. Moreover the functions $\lambda_{i,j}$ descend to the torus symplectic quotients of orbits and hence define integrable systems on the torus quotient of flag manifolds $F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}T$.

Theorem 5.18. *Assume that $F_{\mathbf{k}}(\mathbb{C}^n)$ has the symplectic form corresponding to $a_1\varpi_{k_1} + \cdots + a_m\varpi_{k_m}$ and $F_{\mathbf{l}}(\mathbb{C}^n)$ has the symplectic form corresponding to $b_1\varpi_{l_1} + \cdots + b_m\varpi_{l_m}$ Under the duality isomorphism $\overline{\Psi} : F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}T \to F_{\mathbf{l}}(\mathbb{C}^n)//_{\mathbf{s}}T$ we have*

$$\overline{\Psi}^* \lambda_{i,j} = |\mathbf{b}| - \lambda_{j+1-i,j} = |\mathbf{a}| - \lambda_{i,j}.$$

Proof. It suffices to prove the above formula when $\lambda_{i,j}$ is pulled back to the orbit and so $\overline{\Psi}$ is replaced by Ξ. We have

$$\Xi^* \lambda_{i,j}(A) = \lambda_{i,j}(\Xi(A)) = \lambda_i(B_j(\lambda_{max}(A)I_n) - A)) = \lambda_i(|\mathbf{a}|I_j - B_j(A)).$$

The theorem follows. □

6. Self-duality

Let M be a flag manifold G/P and $M^{opp} = G/P^{opp}$. In this section we investigate the duality map $\overline{\Theta} : M//_{\mathbf{r}}H \to M^{opp}//_{\mathbf{s}}H$ in case P is conjugate to P^{opp} and $\mathbf{r} = \mathbf{s}$. Our main goal is to find the conditions when such a self-duality is trivial i.e. $\overline{\Theta} = Id$. Roughly the following theorems say that self-duality is almost never trivial. For our analysis of the case of $GL_n(\mathbb{C})$ we will take advantage of the solution of the quantum problem in [MTL] although the analysis we give below would work for $GL_n(\mathbb{C})$ as well. Our strategy below will be to first identify those cases with integral $\mathbf{a}$ and $\mathbf{r}$ for which duality is trivial and then show that by scaling by real numbers we obtain all the real cases as well.

6.1. **The existence of good representations.** Recall that we have defined a dominant weight λ (or representation V_λ) to be good if V_λ is self-dual and if there exists N such that the Chevalley involution does not act as a scalar on $V_{N\lambda}[0]$. We will see shortly that this condition on a representation exactly captures nontriviality of the classical duality $\overline{\Theta}$ on the corresponding weight variety. The point of this subsection is to prove that good representations abound. We have

Definition 6.1. *Suppose λ and μ are dominant weights and V_λ and V_μ are the corresponding irreducible representations. Then the Cartan product of V_λ and V_μ is the irreducible representation with highest weight $\lambda + \mu$. There is a canonical surjection $\pi_{\lambda,\mu} : V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$. We will define the N–th Cartan power $C^N V_\lambda$ to be the irreducible representation $V_{N\lambda}$.*

We begin with a very useful lemma - the image of a nonzero *decomposable* vector in the tensor product of two irreducibles in the Cartan product is nonzero.

Lemma 6.2. *Suppose that λ and μ are dominant weights and that $v_1 \in V_\lambda, v_2 \in V_\mu$ are nonzero vectors. Then we have*

$$\pi_{\lambda,\mu}(v_1 \otimes v_2) \neq 0$$

Proof. Use Borel-Weil to interpret v_1 and v_2 as sections s_1 and s_2 of line bundles $\mathcal{L}_\lambda$ and $\mathcal{L}_\mu$ over a flag manifold M. The image $\pi_{\lambda,\mu}(v_1 \otimes v_2)$ then corresponds to the product $s_1 \cdot s_2$ of the two sections under the multiplication map $\Gamma(M, \mathcal{L}_\lambda) \otimes \Gamma(M, \mathcal{L}_\mu) \to \Gamma(M, \mathcal{L}_\lambda \otimes \mathcal{L}_\mu)$. But the product of two nonzero sections is never zero on an irreducible variety. □

We first apply this to

Lemma 6.3. *Suppose that θ does not act as a scalar on $V_{N_0\lambda}[0]$. Then θ does not act as a scalar on $V_{kN_0\lambda}[0]$ for any $k > 0$.*

Proof. Let $k > 0$ be given. It will be convenient to argue in terms of sections. By hypothesis there exists $s \in V_{N_0\lambda}[0]$ such that s is not an eigenvector of θ. We claim

that $s^{\otimes k}$ is not an eigenvector of θ (note that since M is irreducible $s^{\otimes k} \neq 0$). Suppose to the contrary that there exists z with

$$\theta(s^{\otimes k}) = zs^{\otimes k}.$$

Let $z_i, 1 \leq i \leq k$ be the k–th roots of z. Then we have

$$\prod_{i=1}^{k}(\theta(s) - z_i s) = 0.$$

Again because M is irreducible we must have $\theta(s) - z_i s = 0$ for some i. This is a contradiction. □

We now prove that good representations are stable under Cartan product and in fact much more is true.

Theorem 6.4. *Suppose that V_λ and V_μ are self-dual representations and V_λ is good. Then the Cartan product $V_{\lambda+\mu}$ is good.*

Proof. Since V_λ is good there exists N such that the Chevalley involution θ does not act on $V_{N\lambda}[0]$ as a scalar. By the previous lemma θ does not act as a scalar on $V_{kN\lambda}[0], k \geq 1$. Choose k so that $V_{kN\mu}[0] \neq 0$. Now choose a nonzero vector $v_+ \in V_{kN\lambda}[0]$ such that $\theta(v_+) = v_+$ and another nonzero vector $v_- \in V_{kN\lambda}[0]$ such that $\theta(v_-) = -v_-$. The Chevalley involution has either 1 or -1 as an eigenvalue on $V_{kN\mu}[0]$. For convenience assume the former. Let u be an eigenvector belonging to 1. Then the images of $v_+ \otimes u$ and $v_- \otimes u$ in the Cartan product are nonzero by Lemma 6.2 and belong to eigenvalues 1 and -1 respectively. □

6.2. **The branching trick.** In this section we will give the main technique we will use below to prove that certain fundamental representations are good. We will refer to it as the "branching trick". It is (a refinement of) one of the main techniques used in [MTL], see §4.3 and Proposition 4.6.

We begin with the following lemma.

Lemma 6.5. *Let G_1 and G_2 be simple complex Lie groups with $G_1 \subset G_2$. Let λ be a dominant weight for G_2 and μ be a dominant weight for G_1. If V_μ occurs in the restriction of the irreducible representation V_λ of G_2 to G_1 then $V_{N\mu}$ occurs in the restriction of the irreducible representation $V_{N\lambda}$ of G_2 to G_1 for all $N \in \mathbb{N}$.*

Proof. Suppose v is a nonzero vector of weight μ in V_λ which is annihilated by the nilradical of $\mathfrak{g}_1$. Then the image of the vector $v^{\otimes N}$ in $V_{N\lambda}$ is *nonzero* (by Lemma 6.2) has weight $N\mu$ and is again annihilated by the nilradical of $\mathfrak{g}_1$. □

Recall that we have defined a representation V_λ to be good if V_λ is self-dual and for some $N \in \mathbb{N}$ the Chevalley involution does not act on $V_{N\lambda}[0]$ as a scalar. The "branching trick" is then the following

Proposition 6.6. *Suppose that V_λ is an irreducible representation of simple complex Lie group G_2 and that G_1 is a maximal rank subgroup so that the restriction of V_λ to G_1 contains an irreducible summand that is either good or not self-dual. Then V_λ is good.*

Proof. Let H be a Cartan subgroup of G_2 which is contained in G_1. Let θ_1 be a Chevalley involution of G_1 and θ_2 be a Chevalley involution of G_2 such that both involutions stabilize H and consequently act on H by inversion. Since G_1 is the

centralizer of an element of H it follows that θ_2 carries G_1 into itself. Hence by Lemma 2.3, θ_1 and θ_2 are conjugate by an element $Adh, h \in H$, and by Lemma 2.4, they coincide on the zero weight space of any self-dual representation of G_1.

Suppose first that V_μ is good. Since V_μ is good there exists N so that θ_1 does not act as a scalar on $V_{N\mu}[0]$. Let $v \in V_{N\mu}[0]$ satisfy $\theta_1(v) \neq \pm v$. Hence by the above paragraph $\theta_2(v) \neq \pm v$. But by Lemma 6.5, $V_{N\mu} \subset V_\lambda$ whence $V_{N\mu}[0] \subset V_\lambda[0]$ and $v \in V_\lambda[0]$.

Suppose now that V_μ is not self-dual. Choose N such that $N\mu$ is in the root lattice for G_1. Choose any nonzero vector $v \in V_\mu[0]$. Then under the action of θ_1 the vector v goes to into a different irreducible summand (corresponding to a copy of the dual of V_μ) in the restriction of V_λ to G_1. This summand has intersection zero with V_μ by Schur's lemma. Hence once again we have $\theta_1(v) \neq \pm v$. The rest of the argument is identical to that of the previous paragraph. $\square$

6.3. **Quantum versus classical duality.** Our goal in this section is to compare the triviality of quantum and classical self-dualities. Let λ be a dominant weight and M be the corresponding flag manifold. We will identify the weight spaces $V_{N\lambda}[N\mathbf{r}]$ with the N–th graded summand of the spaces of invariant sections. In this section we will assume $\mathbf{r} = 0$ and will identify the map on sections $\widetilde{\Theta}$ with the action of the Chevalley involution θ on the corresponding zero weight space $V_{N\lambda}[0]$. We ask the reader to make the required modifications in the proof to cover the case of $GL_n(\mathbb{C})$ and the action of $\widetilde{\Psi}$ on the graded summands. Recall by Corollary 5.12 this action corresponds with the affine involution

$$\widetilde{\Psi} : V_{N\lambda}[\mathbf{r}] \to V_{N\lambda^\vee}[\mathbf{\Lambda} - \mathbf{r}].$$

We will see below that in this case the self-duality condition forces $\mathbf{r} = (|\mathbf{a}|/2)\varpi_n = (1/2)\mathbf{\Lambda}$.

Theorem 6.7. *Suppose that the symplectic manifold M is self-dual and corresponds to an integral orbit (the orbit of an element $\lambda = \lambda^\vee$ of the weight lattice). Then the classical duality map $\overline{\Theta}$ is nontrivial (i.e.not equal to the identity) on $M//_0H$ (resp. $M//_{\mathbf{r}}H$) if and only if V_λ is good.*

The rest of this section will be devoted to proving the theorem. One direction is easy. If θ acts as a scalar on every summand then it is immediate that $\overline{\Theta}$ is equal to the identity. Indeed we may choose m such that the ring $S^m = \oplus_{k=0}^\infty S^{(km)}$ is generated by elements of degree one (i.e. $k = 1$),see [Bou1],Chapter III, §1.3, Proposition 3. We rename this ring S. By [Do], pg.39, we have an equality of maximal projective spectra

$$Projm(S) = Projm(R).$$

Again θ acts by a scalar on any graded summand of S hence in particular it acts as a scalar on the degree one summand $V = S^{(1)}$. Choose a basis of the degree one elements $V = S^{(1)}$ of S to obtain a projective embedding $F : M//H \to P(V^*)$. We claim that F satisfies

$$F(\overline{\Theta}(x)) = \theta(F(x)).$$

Indeed if we identify the dual of the projective space of the space of sections with the hyperplanes in the space of sections then we have $F(x) = H_x$ where H_x is the

hyperplane of sections that vanish at x, see [GH], page 176. The claim will follow if we show

$$H_{\overline{\Theta}(x)} = \widetilde{\Theta}^{-1}(H_x).$$

But $s(\overline{\Theta}(x)) = 0 \Leftrightarrow \hat{\Theta}^{-1}(s(\overline{\Theta}(x))) \Leftrightarrow \widetilde{\Theta}^{-1}(s)(x) = 0$. The claim follows.

By assumption θ acts as a scalar on V and hence θ acts trivially on $P(V^*)$. Since F is injective we deduce from the equivariance formula immediately above that $\overline{\Theta}$ is the identity.

Remark 6.8. *It is not enough to require that θ act as a scalar on the degree one elements of the original graded ring R because R might not be generated by elements of degree one.*

The rest of this section will be devoted to proving the converse i.e. if there exists N_0 such that θ does not act as a scalar on $V_{N_0\lambda}[0]$ then $\overline{\Theta}$ is not equal to the identity.

Accordingly we assume that θ does not act as a scalar on $V_{N_0\lambda}[0]$. Replace the graded ring R of H–invariant sections by the subring S given by $S = \oplus_{k=0}^{\infty} R^{kN_0}$. By [Do], pg. 39, we have an equality of maximal projective spectra

$$Projm(S) = Projm(R).$$

Also we note that by Lemma 6.3, θ does not act as a scalar on any graded summand of S.

Finally, as before, we may choose m such that the ring $S^m = \oplus_{k=0}^{\infty} S^{(km)}$ is generated by elements of degree one (i.e. $k = 1$). We rename this ring S. Again θ does not act as a scalar on any graded summand of S hence in particular it does not act as a scalar on the degree one summand and again by [Do], pg.39, we have

$$Projm(S) = Projm(R).$$

Now we can complete the proof of the theorem. Let V be the space of degree one elements, $V = S^{(1)}$ of S. We obtain a projective, embedding $F : M//_0H \to P(V^*)$ with $F(x) = H_x$ as above. We have seen that F satisfies

$$F(\overline{\Theta}(x)) = \theta(F(x)).$$

Suppose now for the purpose of contradiction that $\overline{\Theta}$ is the identity. Then for all $x \in M//_0H$ we have $\theta(F(x)) = F(x)$. Let $< ImF >$ denote the smallest projective subspace of $P(V^*)$ containing the image of F. We first check that $< ImF >= P(V^*)$. Indeed, suppose that $< ImF > \subsetneqq P(V^*)$. Then there exists a nonzero element $s \in V$ which pairs to zero with every element of $< ImF >$. But this means that $\forall x \in M//_0H, s(x) = 0$. This is a contradiction.

Now we can prove that θ acts as a scalar on V^* and hence on V. Indeed we have the eigenspace decomposition $V^* = (V^*)^+ \oplus (V^*)^-$. Hence $P(V^*)^+$ and $P(V^*)^-$ are disjoint projective subspaces of $P(V^*)$ with union the fixed-point set of θ on $P(V^*)$. Since M is connected either $ImF \subset P(V^*)^+$ or $ImF \subset P(V^*)^-$. Hence either $P(V^*) =< ImF >= P(V^*)^+$ or $P(V^*) =< ImF >= P(V^*)^-$. In either case we find that θ acts as a scalar on V. This contradicts the assumption that θ does not act as a scalar on any graded summand of S and the theorem is proved.

6.4. **From general flag manifolds to Grassmannians.** In this section we prove the classical analogue of Theorem 6.4 that (in the case that all representations of G are self-dual) will allow us to reduce to the study of $\overline{\Theta}$ from torus quotients of general flag manifolds to "Grassmannians", that is flag manifolds that are quotients G/P where P is *maximal*. In case not all representations are self-dual the result will allow us to reduce to the case of flag manifolds G/P where P is a "next-to-maximal" parabolic (see the treatment of E_6 below).

Let P and Q be parabolic subgroups of G such that $P \subset Q$. Then we have a quotient map $\pi : G/P \to G/Q$. Suppose that the flag manifolds G/P and G/Q are carried into themselves by Θ. We will abbreviate G/P to M and G/Q to N.

Lemma 6.9. *Suppose there exists $z \in N$ such that $\overline{H \cdot \Theta_N(z)} \cap \overline{H \cdot z} \neq \emptyset$. Then for any $w \in \pi^{-1}(z)$ we have*

$$\overline{H \cdot \Theta_M(w)} \cap \overline{H \cdot w} \neq \emptyset.$$

Proof. Suppose $\pi(w) = z$ and $x \in \overline{H \cdot \Theta_M(w)} \cap \overline{H \cdot w}$. Then because π is a closed H–equivariant map we find $z \in \overline{H \cdot \Theta_N(z)}$, a contradiction. $\square$

Let $\overline{\Theta}_M$ resp. $\overline{\Theta}_N$ denote the induced maps on the quotients by H and $Fix(\overline{\Theta}_M)$ resp. $Fix(\overline{\Theta}_N)$ denote their fixed-point sets. By the previous lemma we have

$$\pi^{-1}(Fix(\overline{\Theta}_N)) \subset Fix(\overline{\Theta}_M)).$$

We can now prove the reduction we need.

Proposition 6.10. *Suppose that Θ_N does not induce the identity on $N//_{\mathbf{r}}H$. Then Θ_M does not induce the identity on $M//_{\mathbf{r}}H$.*

Proof. Let $p : N^{ss} \to N//_{\mathbf{r}}H$ be the quotient map. Then, by assumption, $U = p^{-1}(Fix(\overline{\Theta_N})$ is a nonempty Zariski open subset of N whence $V = \pi^{-1}(U)$ is a nonempty Zariski open subset of M. Hence $V \cap M^{ss}$ is nonempty. Let w be a point in this intersection. By the previous lemma we have $\overline{H \cdot \Theta_M(w)} \cap \overline{H \cdot w} \neq \emptyset$ and consequently the image of w in $M//_{\mathbf{r}}$ is not fixed by $\overline{\Theta}_M$. $\square$

Remark 6.11. *The previous Proposition reduces the problem of showing that $\overline{\Theta}$ is nontrivial on a torus quotient of a general flag variety G/P to showing that $\overline{\Theta}$ is nontrivial on the torus quotients of Grassmannians G/Q where Q is a maximal parabolic subgroup containing P.*

6.5. **Splitting the zero level - a nontriviality criterion.** In this subsection we give a useful condition we will use below, see §7.8, to prove that $\overline{\Theta}$ is not equal to the identity for three special cases. We will assume that θ is inner. Consequently Θ takes each Grassmannian into itself.

Suppose that $f : M \to \prod_i^n M_i$ is the inclusion from a flag manifold into a product of Grassmannians and let $f_i : M \to M_i$ be the surjection onto the i–th factor. Let $T \subset \prod_i^n T_i$ be the diagonal inclusion of the maximal compact torus into the product of maximal compact tori and $\mu_i, 1 \leq i \leq n$ the momentum map for the action of T_i on the i–th factor. Then $\mu = \sum_i^n \mu_i$ is the momentum map for T acting on the product.

Lemma 6.12. *Suppose that θ is inner and there exists $x \in M$ such that*

(1) $\mu(f(x)) = 0$.

(2) *For some* $i, 1 \leq i \leq n$ *we have* $\mu_i(f_i(x)) = \mathbf{r}_i \neq 0$.

Then the duality map $\overline{\Theta} : M//_0 H \to M//_0 H$ *is not equal to the identity.*

Proof. Observe that $\mu_i(f_i(\Theta(x))) = -\mathbf{r}_i$ and consequently $f_i(\Theta(x))$ is not in the same T_i orbit as $f_i(x)$. Hence x is not in the same T orbit as $\Theta(x)$. $\square$

6.6. **Self-duality for** $SL_n(\mathbb{C})$**.** For those values of $\mathbf{k}$ and $\mathbf{r}$ such that $\mathbf{k} = \mathbf{l}$ and $\mathbf{r} = \mathbf{s}$ the duality map Ψ is a self-duality. In this section we will prove that except for the case of $Gr_2(\mathbb{C}^4)$ with the symplectic form $2a\varpi_2$ and $\mathbf{r} = \varpi_4$ and one more infinite family of examples (see below)the resulting self-duality maps are not equal to the identity map. We first examine the condition $\mathbf{r} = \mathbf{s}$. We assume that $F_{\mathbf{k}}(\mathbb{C}^n)$ is equipped with the symplectic form induced by embedding it as the orbit of $\sum_i a_i \varpi_i$. Since $s_i = |\mathbf{a}| - r_i$ the following formula is immediate.

Lemma 6.13. $\mathbf{r} = \mathbf{s} \Rightarrow \mathbf{r} = (|\mathbf{a}|/2)\varpi_n$

We need to know that $|\mathbf{a}|/2$ is an integer. This is in fact the case as will be seen in the following lemma.

Let $|\lambda|$ be the sum of the coefficients of λ when λ is expressed in terms of the standard basis. Recall that λ is in the root lattice if and only if $|\lambda|$ is divisible by n. We now have

Lemma 6.14. *Suppose* $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$ *is self-dual. Then*

$$|\lambda|/n = |\mathbf{a}|/2$$

and consequently if λ *is in the root lattice then* $|\mathbf{a}| = \sum_{i=1}^{n-1} a_i$ *is an even integer.*

Proof. Suppose first that n is odd, $n = 2m + 1$. Since $a_i = a_{n-i}$ we have

$$|\lambda| = \sum_{i=1}^{m} a_i i + a_{n-i}(n - i) = (\sum_{i=1}^{m} a_i i)n = (|\mathbf{a}|/2)n.$$

Assume now that $n = 2m$. Then as in the odd case we have

$$|\lambda| = 2m(\sum_{i=1}^{m-1} a_i) + m a_m = 2m(|\mathbf{a}|/2)$$

$\square$

We now recall Theorem 7.2 of [MTL].

Theorem 6.15. *Suppose* λ *is a dominant weight for an irreducible representation of* $SL_n(\mathbb{C})$ *which is in the root lattice. Then the Chevalley involution* θ *of* $SL_n(\mathbb{C})$ *acts as a scalar on the zero weight space* $V_\lambda[0]$ *if and only if* λ *is one of the following*

(1) $\lambda = (a, 0, \dots, 0, -a)$, $a \in \mathbb{N}$.
(2) $\lambda = (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{k})$, $0 \leq k \leq n/2$.
(3) $\lambda = (a, a, -a, -a)$, $a \in \mathbb{N}$.

We want to deduce from this theorem the analogous result for the action of $\widetilde{\Psi}$ on the graded summands of the space of H–invariant sections which we know corresponds to the weight space $V_{N\lambda}[(N/2)|\mathbf{a}|\varpi_n]$. We refer the reader to §2.0.1 for the definition of the action of the Chevalley involution on V_λ and $V_\lambda[0]$ and the definition of the operator Θ_{V_λ}.

We now show we may choose $\widetilde{\Psi}$ for Θ_{V_λ}. This is a consequence of the next lemma which in turn is an immediate consequence of Lemma 5.10.

Lemma 6.16. *Let ρ_λ be the representation of $GL_n(\mathbb{C})$ on V_λ. Assume that V_λ is self-dual. Then we have as representations of $SL_n(\mathbb{C})$*

$$\widetilde{\Psi} \circ \rho_\lambda \circ \widetilde{\Psi} = \rho_\lambda \circ \theta$$

or

$$\widetilde{\Psi} = \Theta_{V_\lambda}.$$

We next need to further modify the above theorem because. we are normalizing the highest weight λ of an irreducible representation of $SL_n(\mathbb{C})$ to have last component zero rather than to have the sum of its components $|\lambda|$ equal to zero. Let H_1 denote the subgroup of H of elements of determinant 1. We note that λ is in the root lattice if and only if $|\lambda|$ is divisible by n and then the zero weight space for H_1 in V_λ coincides with the H weight space $V_\lambda[(|\lambda|/n)\varpi_n]$ $=V_\lambda[(|\mathbf{a}|/2)\varpi_n]$. We now obtain the version of the previous theorem that we need

Corollary 6.17. *Suppose λ is a dominant weight for an irreducible representation of $GL_n(\mathbb{C})$ which is self-dual and in the root lattice. Let M be the flag manifold corresponding to λ and $\mathcal{L}$ be the line bundle over M corresponding to λ. Then the action of $\widetilde{\Psi}$ on the graded component of the graded ring of H–invariant sections of the line bundle $\mathcal{L}$ corresponding to the vector space $V_\lambda[\mathbf{r}]$ is a scalar if and only if either $n = 2$ or λ and $\mathbf{r}$ are one of the following*

1. $\lambda = a\varpi_1 + a\varpi_{n-1}$, $\mathbf{r} = a\varpi_n$,$a \in \mathbb{N}$.
2. $\lambda = \varpi_k + \varpi_{n-k}, 2 \leq k \leq n-2$, $\mathbf{r} = \varpi_n$
3. $n = 4$ *and* $\lambda = 2a\varpi_2$, $\mathbf{r} = a\varpi_4$, $a \in \mathbb{N}$.

We now prove

Theorem 6.18. *Assume that $\mathbf{k}$ and $\mathbf{r}$ satisfy the self-duality conditions $\mathbf{k} = \mathbf{l}$ and $\mathbf{r} = \mathbf{s}$. The self-duality $\overline{\Psi} : F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H \to F_{\mathbf{k}}(\mathbb{C}^n)//_{\mathbf{r}}H$ is equal to the identity if and only if either $n = 2$ or the flag manifold is*

1. $F_{\mathbf{k}}(\mathbb{C}^n) = F_{1,n-1}(\mathbb{C}^n)$ *with the symplectic form* $a\varpi_1 + a\varpi_{n-1}$ *and* $\mathbf{r} = a\varpi_n$.
2. $F_{\mathbf{k}}(\mathbb{C}^n) = Gr_2(\mathbb{C}^4)$ *with the symplectic form* $2a\varpi_2$ *and* $\mathbf{r} = a\varpi_4$.

Proof. The theorem follows from Theorem 6.7. We note that since, by the above theorem, $\widetilde{\Psi}$ acts as a scalar on all graded summands for the two exceptional cases we may apply (the easy direction of) Theorem 6.7 to deduce that $\overline{\Psi}$ is in fact equal to the identity in these two cases. □

6.7. **Self-duality for the isometry groups of bilinear forms.** In this section we assume that M is a flag manifold of a classical group G where G is either a symplectic group or a special orthogonal group. In the case of symplectic groups and odd orthogonal groups -1 is an element of the Weyl group and all dualities are self-dualities. We will use the theory of admissible pairs [La] or [Mu] to construct certain basis elements in the graded summands. However our goal is to find weight zero monomials that are not eigenvectors of θ and as we will see below in order to do this we need only those standard monomials which are products of extremal weight vectors, i.e. correspond to a *trivial* admissible pairs. Hence,we do not need the difficult part of the theory which constructs elements $p(\lambda, \phi)$ for a nontrivial admissible pair λ, ϕ. All we need from the general theory is that the standard

monomials formed from Bruhat chains of extremal weight vectors are linearly independent. We will also use that for the Grassmannians associated to $Sp_{2n}(\mathbb{C})$ and $SO_{2n+1}(\mathbb{C})$ the Bruhat order on the relative Weyl poset W^P coincides with the restriction of the Bruhat order from the ambient linear group [La], page 363, IX and page 365, IX. We will also need the corresponding fact for the Bruhat order on W^P for P the subgroup of $SO_{2n}(\mathbb{C})$ which stabilizes one of the two types of Lagrangian subspaces (so P is miniscule), see [GL], pg. 158.

Our goal is to prove the following theorem.

Theorem 6.19.

(1) *Suppose $G = Sp_{2n}(\mathbb{C})$.Then the duality map $\overline{\Theta}$ is not equal to the identity with the exception of the torus quotients of*
  (a) *The projective space $\mathbb{CP}^{2n-1}$.*
  (b) *The Lagrangian Grassmannian $Gr^0_2(\mathbb{C}^4)$.*

  In both (a) and (b) the map $\overline{\Theta}$ is the identity.

(2) *Suppose now that $G = SO_{2n+1}(\mathbb{C})$. Then the duality map $\overline{\Theta}$ is not equal to the identity with the exception of the torus quotients of*
  (a) *The quadric hypersurface $\mathcal{Q} \subset \mathbb{CP}^{2n}$.*
  (b) *The Lagrangian Grassmannian $Gr^0_2(\mathbb{C}^5)$.*

  In both (a) and (b) the map $\overline{\Theta}$ is the identity.

(3) *Suppose now that $G = SO_{2n}(\mathbb{C})$. Then the duality map $\overline{\Theta}$ is not equal to the identity with the exception of the torus quotients of*
  (a) *The quadric hypersurface $\mathcal{Q} \subset \mathbb{CP}^{2n-1}$.*
  (b) *The isotropic Grassmannian $Gr^0_2(\mathbb{C}^6)$.*
  (c) *The Lagrangian Grassmannians $Gr^0_2(\mathbb{C}^4)^+, Gr^0_2(\mathbb{C}^4)^-, Gr^0_4(\mathbb{C}^8)^+, Gr^0_4(\mathbb{C}^8)^-$*
  (d) *The isotropic flag manifold $F^0_{1,2}(\mathbb{C}^4)$.*

  In (a), (b) (c) and (d) the map $\overline{\Theta}$ is the identity.

7. Proof of theorem 6.19

This section will be devoted to proving the theorem. As explained about we will use the theory of standard monomials (in tableaux form) due to Seshadri and Lakshmibai.

We will begin with a lemma that reduces the proof of the theorem for the classical groups to the case of the Lagrangian Grassmannians or equivalently to the problem of when the last fundamental representation is good.

Lemma 7.1. *If the k–th fundamental representation is good or not self-dual for $G = Sp_{2k}(\mathbb{C})$, resp. $SO_{2k+1}(\mathbb{C})$, resp. $SO_{2k}(\mathbb{C})$ then the k–th fundamental representation is good for $G = Sp_{2n}(\mathbb{C})$, resp. $SO_{2n+1}(\mathbb{C})$, resp. $SO_{2n}(\mathbb{C})$ with $n > k$.*

Proof. First observe that for $Sp_{2n}(\mathbb{C})$ the fundamental representations are the primitive exterior representations $\bigwedge^k_0(\mathbb{C}^{2n})$ of the standard representation and for the orthogonal groups they are either the exterior powers of the standard representation or a *Spin* representation. In this later case the Cartan square is an exterior power of the standard representation (or else contained in it as the subspace fixed by the complex Hodge star).

We claim we have following formulas for branching to the maximal rank subgroups $Sp_{2k}(\mathbb{C}) \times Sp_{2(n-k)}(\mathbb{C})$, resp. $SO_{2k+1}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$, resp. $SO_{2k}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$.

(1) $\bigwedge_0^k(\mathbb{C}^{2n})|Sp_{2k}(\mathbb{C}) \times Sp_{2(n-k)}(\mathbb{C})$ contains $\bigwedge_0^k(\mathbb{C}^{2k}) \boxtimes \bigwedge^0(\mathbb{C}^{2n-2k})$ as representations of $Sp_{2k}(\mathbb{C}) \times Sp_{n-k}(\mathbb{C})$.
(2) $\bigwedge^k(\mathbb{C}^{2n+1})|SO_{2k+1}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$ contains $\bigwedge^k(\mathbb{C}^{2k+1}) \boxtimes \bigwedge^0(\mathbb{C}^{2n-2k})$ as representations of $SO_{2k+1}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$.
(3) $\bigwedge^k(\mathbb{C}^{2n})|SO_{2k}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$ contains both $\bigwedge^k(\mathbb{C}^{2k})_+ \boxtimes \bigwedge^0(\mathbb{C}^{2n-2k})$ and $\bigwedge^k(\mathbb{C}^{2k})_- \boxtimes \bigwedge^0(\mathbb{C}^{2n-2k})$ as representations of $SO_{2k}(\mathbb{C}) \times SO_{2(n-k)}(\mathbb{C})$.

These formulas in turn are an immediate consequence of the formula that the standard representation restricts to the direct sum of the two representations obtained from the standard representation from one factor tensored with the trivial representation from the other factor together with the usual formula for the k–th exterior power of a direct sum. Indeed to prove (1) we first note that the analogue holds for the full exterior power $\bigwedge^k(\mathbb{C}^{2n})$. But in general we have $\bigwedge_0^k(\mathbb{C}^{2n}) = \oplus_{j=0}^k \bigwedge_0^j(\mathbb{C}^{2i}) \otimes \bigwedge_0^{k-j}(\mathbb{C}^{(2k-2i)})$ and the first formula follows.

The second formula is follows immediately from the usual formula for the exterior power of a direct sum. The third formula follows from the formula for restricting an exterior power together with the fact that $\bigwedge^k(\mathbb{C}^{2k})$ is the direct sum of $\bigwedge^k(\mathbb{C}^{2k})_+$ and $\bigwedge^k(\mathbb{C}^{2k})_-$.

The lemma now follows from Proposition 6.6 and the observation $C^N(U \boxtimes V) = C^N(U) \boxtimes C^N(V)$ where $U \boxtimes V$ is the outer tensor product of the irreducible representations U of G_1 and V of G_2 (here $C^N(W)$ denotes the N–th Cartan power of W). □

7.1. **The second fundamental representation for the symplectic and orthogonal groups.** Before beginning our study of the last fundamental representation(s) for the classical groups we deal with the case of the Grassmannians $Gr_2^0(\mathbb{C}^n)$.

Lemma 7.2. *Let V_{ϖ_2} denote the irreducible representation of either $Sp_{2n}(\mathbb{C})$ or $SO_{2n+1}(\mathbb{C})$ with highest weight ϖ_2. Then*
(1) *V_{ϖ_2} is good for $Sp_{2n}(\mathbb{C})$ provided $n \geq 3$.*
(2) *V_{ϖ_2} is good for $SO_{2n+1}(\mathbb{C})$ provided $n \geq 3$.*
(3) *V_{ϖ_2} is good for $SO_{2n}(\mathbb{C})$ provided $n \geq 4$.*

Moreover in all three cases the Chevalley involution does not act as a scalar on $V_{3\varpi_2}[0]$.

Proof. We first give a proof using standard monomials valid for the symplectic and odd orthogonal cases. The weight zero cubic monomial m_I in the Plücker coordinates given by $m_I = X_{12}X_{3\overline{2}}X_{\overline{31}}$ is not an eigenvector of J. Hence J has eigenvalues of both signs on the third graded summand of the homogeneous coordinate ring of $Gr_2^0(\mathbb{C}^n)//_0H$.

We now deal with the case of $SO_{2n}(\mathbb{C})$. First we treat the case of $SO_8(\mathbb{C})$ by branching the third Cartan power to the maximal subgroup of maximal rank $(SL_2(\mathbb{C}))^4$. Each zero weight summand in the restriction is the Cartan product $S^{k_1}(\mathbb{C}^2) \boxtimes S^{k_2}(\mathbb{C}^2) \boxtimes S^{k_3}(\mathbb{C}^2) \boxtimes S^{k_4}(\mathbb{C}^2)$. Here $S^k(\mathbb{C}^2)$ denotes the k–th symmetric power of the standard representation. The zero weight space of this summand is nonzero if and only if each k_i is even. The Chevalley involution acts on this summand as the outer tensor product of the Chevalley involution in each factor and hence each summand is an eigenvector of Chevalley (because the zero weight space of each summand is one dimensional). The sign on the i–th factor depends

whether k_i is congruent to 0 or 2 modulo 4 (it is $+1$ in the first case and -1 in the second). Using LiE we find the -1 eigenspace is large and the $+1$ eigenspace is one-dimensional.

To do the case of general n we branch to the maximal subgroup of maximal rank $SO_{2n-2}(\mathbb{C}) \times SO_2(\mathbb{C})$. By the exterior product of a direct sum formula we have $\bigwedge^2(\mathbb{C}^{2n})|SO_{2n-2}(\mathbb{C}) \times SO_2(\mathbb{C})$ contains $\bigwedge^2(\mathbb{C}^{2n-2}) \boxtimes \bigwedge^0(\mathbb{C}^2)$ as a summand. But this summand is good by induction. □

Remark 7.3. *The representation V_{ϖ_2} is not good for $SO_6(\mathbb{C})$. Indeed the second exterior power $\bigwedge^2(\mathbb{C}^6)$ is not a fundamental representation. It is the Cartan product of the two spin representations, it has highest weight $(1,1,0) = \varpi_2 + \varpi_3$. Under the isomorphism between $Spin_6(\mathbb{C})$ and $SL_4(\mathbb{C})$ it corresponds to the sum of the first and third fundamental representations which in turn corresponds to the exceptional case of the flag manifold $F_{1,3}(\mathbb{C}^4)$. Thus the Chevalley involution acts as a scalar on the zero weight spaces of all Cartan powers of $\bigwedge^2(\mathbb{C}^6)$.*

7.2. **The symplectic group $Sp_{2n}(\mathbb{C})$.**

In this subsection we will prove

Lemma 7.4. *The representation of $Sp_{2n}(\mathbb{C})$ with highest weight ϖ_n is good if $n \geq 3$.*

Note that $J(e_i) = \epsilon e_{\overline{i}}$ where $\overline{i} = 2n+1-i$ and $\epsilon = -1$ if $1 \leq i \leq n$ and $+1$ otherwise. Each column of the tableaux below represents the wedge of the coordinate vectors corresponding to the entries in that column. According J acts on the basis vector represented by the tableau by changing each entry to its bar and multiplying by a sign which will not be important to us here.

The rest of this subsection will be devoted to proving the lemma.

Consider the case of $Gr_3^0(\mathbb{C}^6)$, the space of isotropic three dimensional subspaces of $\mathbb{C}^6$. Let

$$\alpha = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 4 & 5 \\ \hline 3 & 5 & 6 & 6 \\ \hline \end{array} \qquad \beta = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 5 & 6 & 6 \\ \hline \end{array}$$

The sections α and β are standard basis vectors (i.e. correspond to standard monomials) of the irreducible representation $V_{4\varpi_3}$ since the columns correspond to *extremal* weights of the third exterior power of the standard representation (i.e. they index isotropic coordinate planes) and they are increasing in the Bruhat order. They are in the 0–weight space $V_{4\varpi_3}[0]$, since the indices 1 through 6 appear exactly twice each (more generally, an index i needs to appear with the same frequency as its complement $\overline{i} = 2n+1-i$.) The Chevalley involution θ maps α to β. Since α and β represent standard monomials, the sections corresponding to α and β are linearly independent. Hence θ does not act as a scalar on $V_{4\varpi_3}[0]$.

We now construct analogous sections α_n and β_n of the irreducible representation $V_{4\varpi_n}[0]$ of $Sp_{2n}(\mathbb{C})$ for all $n \geq 3$ by induction. Suppose that $n \geq 4$ and $\alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$ have already been constructed. Let α_n have an n by 4 diagram, and let $\alpha_n(1,1) = \alpha_n(1,2) = 1$, $\alpha_n(n,3) = \alpha_n(n,4) = 2n$. For $j = 1,2$, and $i \geq 2$ let $\alpha_n(i,j) = \alpha_{n-1}(i-1,j)+1$, and for $j = 3,4$ and $i \leq n-1$ let $\alpha_n(i,j) = \alpha_{n-1}(i,j)+1$. In other words, to get α_n, first add 1 to all the entries of α_{n-1}. Then slide the

first two columns down one level, and put in two 1's and two $2n$'s in the remaining empty slots. Here is an example to get α_4 from α_3.

$$\alpha_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 4 & 5 \\ \hline 3 & 5 & 6 & 6 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 4 \\ \hline 3 & 5 & 5 & 6 \\ \hline 4 & 6 & 7 & 7 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline & & 3 & 4 \\ \hline 2 & 2 & 5 & 6 \\ \hline 3 & 5 & 7 & 7 \\ \hline 4 & 6 & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 & 6 \\ \hline 3 & 5 & 7 & 7 \\ \hline 4 & 6 & 8 & 8 \\ \hline \end{array} = \alpha_4$$

Since the columns of α_{n-1} are isotropic, no column of α_{n-1} contains both an index i and its complement $\overline{i} = (2n-1) - i$. Note that $\overline{i+1} = 2n + 1 - (i+1) = ((2n-1)-i)+1 = \overline{i}+1$. Therefore the columns of α_n are isotropic as well. Sliding the first two columns down preserves the property that rows are weakly increasing, and the weight of α_n is 0 since all indices occur twice.

The section β_n is constructed from β_{n-1} in the same way as α_n is constructed from α_{n-1}. The Chevalley involution takes α_n to β_n. Since β_n is standard it is independent of α_n and consequently the Chevalley involution has eigenvalues $+1$ and -1 on degree 4 weight 0 sections. The lemma is now proved.

We have now shown that $\overline{\Theta}$ is nontrivial on the torus quotients of the symplectic Grassmannians $Gr_n^0(C^{2m})//_0H$ for all $n \geq 3$.

7.3. **The orthogonal group $SO_{2n+1}(\mathbb{C})$.**

In this subsection we will prove

Lemma 7.5. *The representation of $SO_{2n+1}(\mathbb{C})$ with highest weight ϖ_n is good if $n \geq 3$.*

The construction is similar to that of the symplectic case, as one might expect since the Weyl groups are the same. The difference is that the middle index $n+1$ may not appear in the tableau. To get α_n for the orthogonal group from the α_n for the symplectic group, simply add one to each index which is greater than or equal to $n+1$. For example, when $n = 3$,

$$\alpha_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 5 & 5 & 6 \\ \hline 3 & 6 & 7 & 7 \\ \hline \end{array} \qquad \beta_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 3 & 6 \\ \hline 5 & 6 & 7 & 7 \\ \hline \end{array}$$

7.4. **The orthogonal groups $SO_{4n}(\mathbb{C})$.**

In this subsection we will prove

Lemma 7.6. *The irreducible representations of $SO_{4n}(\mathbb{C})$ with highest weights ϖ_{2n-1} or ϖ_{2n} are good if $n \geq 3$.*

The Grassmannian of isotropic n-dimensional spaces in $\mathbb{C}^{2n}$ has two components, $Gr_n^0(\mathbb{C}^{2n})^+$ and $Gr_n^0(\mathbb{C}^{2n})^-$. The corresponding representations are the two spin representations Δ_{2n}^+ and Δ_{2n}^- and are miniscule. The weights of each representation lie in a single Weyl group orbit and consequently must have the same parity in the number of negative signs, since any Weyl group element must negate an even number of components. Without loss of generality we will treat the case of $Gr_n^0(\mathbb{C}^{2n})^+$. The (extremal) standard monomials correspond to tableaux that have columns representing weights which are all in the same Weyl orbit, and increasing in the Bruhat order induced from $SL_{2n}(\mathbb{C})$, see [GL], page 158, for a description of the Bruhat poset.

We now show that the Chevalley involution does not act as a scalar for n even, $n \geq 6$. To prove this we begin with $Gr_6^0(\mathbb{C}^{12})$. Let

$$\alpha_6 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 3 & 7 & 7 & 9 \\ \hline 4 & 8 & 8 & 10 \\ \hline 5 & 9 & 10 & 11 \\ \hline 6 & 11 & 12 & 12 \\ \hline \end{array} \qquad \beta_6 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 7 \\ \hline 2 & 3 & 4 & 8 \\ \hline 3 & 5 & 5 & 9 \\ \hline 4 & 6 & 6 & 10 \\ \hline 7 & 9 & 10 & 11 \\ \hline 8 & 11 & 12 & 12 \\ \hline \end{array}$$

The reader will observe that α_6 and β_6 satisfy that each column has even parity, and so they define standard basis elements of the fourth Cartan power of the even spin representation Δ_{12}^{+} of $Spin(12)$. Furthermore, α_6 is mapped by θ to $\beta_6 \neq \alpha_6$.

We construct α_{2k} and β_{2k} for $k \geq 3$. To get α_{2k} from α_{2k-2}, first add 2 to each entry, then slide the first two columns down two levels, and put in two each of $1, 2, 4k-1, 4k$ in the appropriate positions.

For example,

$$\alpha_6 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 3 & 7 & 7 & 9 \\ \hline 4 & 8 & 8 & 10 \\ \hline 5 & 9 & 10 & 11 \\ \hline 6 & 11 & 12 & 12 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline & & 4 & 7 \\ \hline & & 6 & 8 \\ \hline 3 & 3 & 9 & 11 \\ \hline 4 & 5 & 10 & 12 \\ \hline 5 & 9 & 12 & 13 \\ \hline 6 & 10 & 14 & 14 \\ \hline 7 & 11 & & \\ \hline 8 & 13 & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 4 & 7 \\ \hline 2 & 2 & 6 & 8 \\ \hline 3 & 3 & 9 & 11 \\ \hline 4 & 5 & 10 & 12 \\ \hline 5 & 9 & 12 & 13 \\ \hline 6 & 10 & 14 & 14 \\ \hline 7 & 11 & 15 & 15 \\ \hline 8 & 13 & 16 & 16 \\ \hline \end{array} = \alpha_8$$

The parity is still even since adding 2 does not change the parity as you go from $2k-2$ to $2k$, and adding the indices $1, 2$ does not affect the number of negative signs of the weights associated to the first two columns. Adding $4k-1, 4k$ to the last two columns adds two negative signs, and thus parity remains even.

The section β_{2k} is formed in the same manner from the β_{2k-2}. Hence the Chevalley involution is non-trivial in the higher dimensions.

We need two more lemmas to take care of some missing cases.

Lemma 7.7. *The fundamental representation $V_{\varpi_{2n-1}}$ for $SO_{4n}(\mathbb{C})$ is good provided $n \geq 2$.*

Proof. We have $\bigwedge^{2n-1}(\mathbb{C}^{4n})|SO_{4n-2}(\mathbb{C}) \times SO_2(\mathbb{C})$ contains the nonself-dual representation $\bigwedge^{2n-1}(\mathbb{C}^{4n-2})_+ \boxtimes \bigwedge^0(\mathbb{C}^2)$. □

We also need

Lemma 7.8. *The fundamental representation V_{ϖ_4} for $SO_{4n}(\mathbb{C})$ is good provided $n \geq 3$.*

Proof. We have $\bigwedge^4(\mathbb{C}^{4n})|SO_{4n-2}(\mathbb{C}) \times SO_2(\mathbb{C})$ contains the representation $\bigwedge^2(\mathbb{C}^{4n-2}) \boxtimes \bigwedge^2(\mathbb{C}^2)$. □

All other fundamental representations for $SO_{4n}(\mathbb{C})$ follow from Lemma 7.6.

We have proved the following

Proposition 7.9. *All the fundamental representations except the first of $SO_{4n}(\mathbb{C})$ are good provided $n \geq 3$.*

7.5. **The orthogonal groups $SO_{4n+2}(\mathbb{C})$.** In the case of $SO_{4n+2}(\mathbb{C})$ the last two fundamental representations are not self-dual. The Cartan product of the last two fundamental representations is the exterior power $\bigwedge^{2n}(\mathbb{C}^{4n+2})$.

Every self-dual representation is a Cartan product of Cartan powers of the first $2n-1$ fundamental representations together with the Cartan powers of $\bigwedge^{2n}(\mathbb{C}^{4n+2})$. By Theorem 6.4 we will be done once we prove

Lemma 7.10. *The representation $\bigwedge^{2n}(\mathbb{C}^{4n+2}) = V_{\varpi_{2n}+\varpi_{2n+1}}$ is good if $n \geq 2$.*

Proof. We have $\bigwedge^{2n}(\mathbb{C}^{4n+2})|SO_{4n}(\mathbb{C}) \times SO_2(\mathbb{C})$ contains $\bigwedge^{2n-2}(\mathbb{C}^{4n}) \boxtimes \bigwedge^2(\mathbb{C}^2)$. The first factor is good provided $n \geq 2$. $\square$

We have concluded our analysis of the even special orthogonal groups $SO_{2n}(\mathbb{C})$.

Proposition 7.11. *All of the self-dual representations except the Cartan powers of the standard representation of $SO_{2n}(\mathbb{C})$ are good provided $n \geq 5$.*

7.6. **The exceptional cases for $Sp_{2n}(\mathbb{C})$ and $SO_{2n+1}(\mathbb{C})$.** We first prove that $\overline{\Theta}$ is trivial for torus quotients of the space of lines in the symplectic vector space $\mathbb{C}^{2n}$. Let $x_i, 1 \leq i \leq 2n$ be the linear coordinates relative to an adapted basis chosen as before so $(e_i, e_{2n+1-i}) = 1$ and all other symplectic products are zero. We let $\overline{i} = 2n+1-i, 1 \leq i \leq 2n$. It is then apparent that in any H–invariant monomial x_I in the coordinates x_i the indices i and $\overline{i}$ must appear the same number of times and consequently x_I is invariant under θ. An analogous argument takes care of the torus quotients of the quadrics $\mathcal{Q}$. We leave to the reader the task of checking that any H–invariant monomial in the Plücker coordinates for $Gr_2^0(\mathbb{C}^4)$ (the symplectic case) and $Gr_2^0(\mathbb{C}^5)$ (the orthogonal case) is invariant under θ.

Finally it remains to treat the case of the flag manifold of lines and planes in $\mathbb{C}^4$. We will do this at the end of §7.8 for the case of general *real* parameters.

7.7. **The exceptional cases for $SO_{2n}(\mathbb{C})$.** We prove that $\overline{\Theta}$ acts trivially on the torus quotients of the quadrics $\mathcal{Q}$ with a symplectic form corresponding to an integral orbit in the same way as we did for torus quotients of the space of lines in the symplectic vector space $\mathbb{C}^{2n}$.

Also $\overline{\Theta}$ acts trivially on the torus quotients of $Gr_2^0(\mathbb{C}^6)$ because under the isomorphism between $SO_6(\mathbb{C})$ and $SL_4(\mathbb{C})$ the Grassmannian $Gr_2^0(\mathbb{C}^6)$ corresponds to the flag manifold $F_{1,3}(\mathbb{C}^4)$. Similarly $\overline{\Theta}$ does not act trivially on the torus quotient of $F_{1,2}^0(\mathbb{C}^6)$ because under the above isomorphism $F_{1,2}^0(\mathbb{C}^6)$ corresponds to the *full* flag manifold $F_{1,2,3}(\mathbb{C}^4)$.

Next we explain why $\overline{\Theta}$ is trivial on the torus quotients of the Lagrangian spaces $Gr_2^0(\mathbb{C}^4)^+$, $Gr_2^0(\mathbb{C}^4)^-$, $Gr_4^0(\mathbb{C}^8)^+$, and $Gr_4^0(\mathbb{C}^8)^-$. It is easy to check (for example by using the Spin representation) that $Gr_2^0(\mathbb{C}^4)^+$, and $Gr_2^0(\mathbb{C}^4)^-$ are isomorphic to $\mathbb{CP}^1$ and the corresponding torus quotients are points so it is clear that $\overline{\Theta}$ is trivial for these two cases. As for the cases of $Gr_4^0(\mathbb{C}^8)^+$, and $Gr_4^0(\mathbb{C}^8)^-$ it follows from the triality isomorphism, [FuHa], §20.3, that each of these two flag manifolds is isomorphic to the quadric $\mathcal{Q}_6$ by an isomorphism that is torus equivariant (though perhaps with a different but equivalent action) and conjugates the Chevalley involution to a new involution that still acts on the torus by inversion. Hence by Lemma 2.3 the new involution is conjugate to the Chevalley involution by an element Adh and induces the same map as the Chevalley involution on any torus quotient. Since we

have seen that $\overline{\Theta}$ is trivial on $\mathcal{Q}_6//_0H$ it follows that $\overline{\Theta}$ is trivial on $Gr_4^0(\mathbb{C}^8)^+//_0H$ and $Gr_4^0(\mathbb{C}^8)^-//_0H$.

We have now dealt with the cases of $Gr_n^0(\mathbb{C}^{2n})^+$ and $Gr_n^0(\mathbb{C}^{2n})^-$ for $n=2$ and $n=4$. The torus quotient of the flag manifold $F_{1,2}^0(\mathbb{C}^4)$ is equal to a point so it is trivial that $\overline{\Theta}$ is the identity for this case.

It remains to prove that $\overline{\Theta}$ is not equal to the identity for the quotients$F_{1,2}^0(\mathbb{C}^4)$ and $F_{1,4}^0(\mathbb{C}^8)^{\pm}//_0H$. We will prove this in the next section.

7.8. **From integral parameters to general parameters.** We first observe that by Proposition 6.10 it suffices (except for a small number of examples) to promote the nontriviality results obtained above from integral parameters to general parameters for *Grassmannians*. Here there is no problem. We use the "scaling trick". Namely if we scale the symplectic form by a real number c (thereby multiplying a and $\mathbf{r}$ by c we do not change the torus quotient and we do not change $\overline{\Theta}$. To be precise suppose the symplectic form is induced by embedding M into $\mathfrak{k}^*$ as the orbit $\mathcal{O}_\lambda$. Let m_c be the automorphism of $\mathfrak{k}^*$ given by multiplication by c. Then m_c is T–equivariant and we have the following diagram

$$\begin{array}{ccc} \mathcal{O}_\lambda//_{\mathbf{r}}T & \xrightarrow{m_c} & \mathcal{O}_{c\lambda}//_{\mathbf{r}}T \\ {\scriptstyle\overline{\Theta}}\Big\uparrow & & \Big\uparrow{\scriptstyle\overline{\Theta}} \\ \mathcal{O}_\lambda//_{\mathbf{r}}T & \xrightarrow{m_c} & \mathcal{O}_{c\lambda}//_{\mathbf{r}}T. \end{array}$$

Thus $\overline{\Theta}$ is either trivial for all c or nontrivial for all c.

This takes care of all the Grassmannian cases (i.e. the cases where P is maximal). Also by using the "scaling trick" we may promote all of the above results from integral parameters to rational parameters. By continuity this allows us to promote all the cases where we have proved that $\overline{\Theta}$ is trivial to the general case.

It remains to prove that $\overline{\Theta}$ is not equal to the identity on the torus quotients of $F_{1,2}^0(\mathbb{C}^4)$ and the two flag manifolds $F_{1,4}^0(\mathbb{C}^8)^+$ and $F_{1,4}^0(\mathbb{C}^8)^-$. In all three cases there is a *two* real parameter family of symplectic forms. Because we have a two parameter family the scaling trick does not suffice to extend our nontriviality result from integral parameters to general parameters. We will give complete details in the second case and third cases and give the main point for the easier first case. First we claim it suffices to treat the case of $F_{1,4}^0(\mathbb{C}^8)^+$. Indeed the matrix in $O(8)$ which interchanges the fourth and fifth standard basis vectors and leaves all the other basis vectors fixed interchanges $F_{1,4}^0(\mathbb{C}^8)^+$ and $F_{1,4}^0(\mathbb{C}^8)^-$, commutes with Θ and normalizes the torus. Hence $\overline{\Theta}$ is the identity on $F_{1,4}^0(\mathbb{C}^8)^+//_0T$ if and only if it is the identity on $F_{1,4}^0(\mathbb{C}^8)^-//_0T$. We now prove

Lemma 7.12. *The map $\overline{\Theta}$ is not equal to the identity on $F_{1,4}^0(\mathbb{C}^8)^+//_0T$ for any of the symplectic quotients for the symplectic forms corresponding to the orbits of $a\varpi_1 + b\varpi_4 = (a+(b/2), b/2, b/2, b/2)$.*

Proof. We will apply Lemma 6.12 to deduce that $\overline{\Theta}$ is not equal to the identity.

Let $St_4^0(\mathbb{C}^8)$ denote the submanifold of the Stiefel manifold of 8 by 4 complex matrices with columns which are orthonormal for the standard hermitian form such that the columns span a subspace which is Lagrangian for the bilinear form $(x,y) = \sum_{i=1}^8 x_i y_{9-i}$. Let $\pi_1 : St_4^0(\mathbb{C}^8) \to Gr_1^0(\mathbb{C}^8)$ be the map sending a matrix to the span of its first column, and let $\pi_4 : St_4^0(\mathbb{C}^8) \to Gr_4^0(\mathbb{C}^8)$ take A to $Im(A)$.

Let $\pi_{1,4} : St_4^0(\mathbb{C}^8) \to F_{1,4}^0(\mathbb{C}^8)$ be given by $\pi_{1,4}(A) = (\pi_1(A), \pi_4(A))$. Let ω_1 be the symplectic form corresponding to ϖ_1 for the isotropic Grassmannian $Gr_1^0(\mathbb{C}^8)$, and let ω_4 correspond to $2\varpi_4$ for $Gr_4^0(\mathbb{C}^8)$. A momentum mapping μ_1 for $Gr_1^0(\mathbb{C}^8)$ is given by $\mu_1(\pi_1(A)) = (|a_{11}|^2 - |a_{81}|^2, |a_{21}|^2 - |a_{71}|^2, |a_{31}|^2 - |a_{61}|^2, |a_{41}|^2 - |a_{51}|^2)$. We claim that a momentum mapping for $Gr_4^0(\mathbb{C}^8)$ is given by $\mu_4(\pi_4(A)) = (|r_1|^2 - |r_8|^2, |r_2|^2 - |r_7|^2, |r_3|^2 - |r_6|^2, |r_4|^2 - |r_5|^2)$, where r_i is the i-th row vector of A and $|r_i|^2$ is the length of the i–th row for the standard Hermitian form on $\mathbb{C}^4$. Indeed since the columns of A are orthonormal for the standard Hermitian form on $\mathbb{C}^8$ it follows that the momentum map ν for the action of the diagonal torus in $U(8)$ is given by $\nu(A) = (|r_1|^2, |r_2|^2, \cdots, |r_8|^2)$. The torus T for $SO(8)$ is embedded in the torus for $U(8)$ as the set of diagonal matrices such that $z_i = z_{9-i}^{-1}, 1 \le i \le 8$. Since the momentum map for T is the orthogonal projection onto the Lie algebra of T the claim follows. Thus $\mu_{1,4}^{a,b} = a\mu_1 + b\mu_4$ is a momentum map for the natural symplectic embedding of $F_{1,4}^0(\mathbb{C}^8)^+$ into $Gr_1^0(\mathbb{C}^8) \times Gr_4^0(\mathbb{C}^8)^+$ for the symplectic form $a\omega_1 + b\omega_4$ on the product.

Let

$$A = \begin{pmatrix} +\alpha & +\alpha & +\gamma & +\gamma \\ +\alpha & +\alpha & -\gamma & -\gamma \\ +\alpha & -\alpha & +\gamma & -\gamma \\ +\alpha & -\alpha & -\gamma & +\gamma \\ +\beta & +\beta & +\delta & +\delta \\ +\beta & +\beta & -\delta & -\delta \\ -\beta & +\beta & -\delta & +\delta \\ -\beta & +\beta & +\delta & -\delta \end{pmatrix}$$

First we claim that the flag corresponding to A as above is isotropic and belongs to $F_{1,4}^0(\mathbb{C}^8)^+$. Indeed the linear functional x_I on $\bigwedge^4(\mathbb{C}^8)$ obtained by wedging together the first four elements of the basis dual to the standard basis of $\mathbb{C}^8$ takes value $-16\alpha^2\gamma^2$ on A (the determinant of the upper four by four block). But x_I is fixed by the Hodge star and consequently takes the value zero on any element of $F_{1,4}^0(\mathbb{C}^8)^-$.

Next note that the columns of $A \in St_4^0(\mathbb{C}^8)$ and $\mu_{1,4}^{a,b}(\pi_{1,4}(A)) = 0$ if $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfy the following equations:

(1) $4\alpha^2 + 4\beta^2 = 1, 4\gamma^2 + 4\delta^2 = 1$ (unit length)
(2) $(a + 2b)(\alpha^2 - \beta^2) + 2b(\gamma^2 - \delta^2) = 0$ (momentum 0)

For small enough $t \in \mathbb{R}$, solutions are given by

$$\alpha(t) = \sqrt{1/8 + 2bt} \tag{4}$$

$$\beta(t) = \sqrt{1/8 - 2bt} \tag{5}$$

$$\gamma(t) = \sqrt{1/8 - (a + 2b)t} \tag{6}$$

$$\delta(t) = \sqrt{1/8 + (a + 2b)t} \tag{7}$$

For $t \neq 0$, $\alpha(t)^2 \neq \beta(t)^2$, and hence $a\mu_1(\pi_1(A(t))) \neq 0$. Hence, $[A(t)] \in F_{1,4}^0(\mathbb{C}^8)^+//_0 H$ is not fixed by $\overline{\Theta}$ by Lemma 6.12.

□

For the symplectic flag manifold $F_{1,2}^0(\mathbb{C}^4)$, the reader will give an analogous (but easier) argument using the matrix

$$A = \begin{pmatrix} \alpha & \gamma \\ \alpha & -\gamma \\ \beta & \delta \\ \beta & -\delta \end{pmatrix}$$

8. The exceptional groups

In this section we will prove that the self-duality map $\overline{\Theta}$ is never equal to the identity on a torus quotient of a flag manifold of an exceptional group.

We will prove the following theorem below using the branching trick

Theorem 8.1. *Let G be an exceptional Lie group and V_λ be a self-dual representation of G. Then V_λ is good.*

By Theorem 6.7 we then obtain

Corollary 8.2. *Let M be an integral self-dual flag manifold associated to an exceptional group. Then the self-duality map $\overline{\Theta}$ on $M//_0H$ is not equal to the identity.*

We then pass from integral parameters to general parameters using the scaling trick.

8.1. **The group G_2.** See [MTL] Theorem 1.7. By branching to $SL_3(\mathbb{C})$ one finds that both the fundamental representations of G_2 are good and hence by Theorem 6.4 all representations are good.

8.2. **The group F_4.** We restrict the fundamental representations $V_i, 1 \leq i \leq 4$ to the subgroup $Spin(9)$.The only bad representations for $Spin(9)$ are the Cartan powers of the first fundamental representation. We check by Lie that the restriction of every fundamental representation contains a good irreducible summand.

8.3. **The group E_6.** We will use the notation of [Bou2], page 261.

Lemma 8.3. *Any self-dual highest weight λ may be written*

$$\lambda = a(\varpi_1 + \varpi_6) + b(\varpi_3 + \varpi_5) + c\varpi_2 + d\varpi_4.$$

As a consequence a self-dual irreducible representation V_λ is a quotient of a tensor product of Cartan powers of $V_{\varpi_1+\varpi_6}$,$V_{\varpi_3+\varpi_5}$, V_{ϖ_2} and V_{ϖ_4}.

We restrict the four basic representations $V_{\varpi_1+\varpi_6}$, $V_{\varpi_3+\varpi_5}$, V_{ϖ_2} and V_{ϖ_4} to the maximal rank subgroup $SL_5(\mathbb{C}) \times SL_2(\mathbb{C})$. We find using LiE that the restriction of each of the four representations contains either a nonself-dual or a good irreducible summand. Hence each of these representations is good and hence by Theorem 6.4 any quotient of a tensor product involving a Cartan power of one of the four basic representations is good. Hence any self-dual representation is good.

8.4. **The group E_7.** We restrict the fundamental representations $V_i, 1 \leq i \leq 7$ to $SL_8(\mathbb{C})$. The only self-dual representations for $SL_8(\mathbb{C})$ are the Cartan powers of $\varpi_1 + \varpi_7$. We again verify by LiE that the restriction of each V_i all contain either a good representation or a nonself-dual irreducible summand.

8.5. **The group E_8.** We restrict the fundamental representations to $SL_9(\mathbb{C})$. We again check by LiE that at least one good representation occurs in the restriction of each fundamental representation of E_8.

9. Further questions

It appears that the duality map Θ preserves almost every important structure connected with the Grassmannian. We list some that need to be further investigated.

9.1. **Toric Degenerations of Torus Quotients of Flag Manifolds.** The duality map carries the toric space that is the degeneration of $G_{m+1}(\mathbb{C}^n)//_{\mathbf{r}}H$ to the toric space that is the degeneration of $G_{n-m-1}(\mathbb{C}^n)//_{\mathbf{s}}H$ in the construction of P. Foth and Yi Hu in [FH] since it permutes the Gelfand-Tsetlin Hamiltonians, see §5.6.1. At the moment we have not yet proved that the duality map can be extended to a map of total spaces of the degeneration. There are a number of other toric degenerations of flag manifolds, which induce toric degenerations of torus quotients of flag manifolds, see for example [GL], Chapter 11, where it remains to extend the duality map to the total space of the degenerations.

9.2. **Gel'fand Hypergeometric Functions.** In the 1980's Gel'fand and his collaborators created a theory of hypergeometric functions on Grassmannians generalizing the classical theory defined on the moduli spaces $\mathcal{M}_r(\mathbb{CP}^1)$, see [DM]. There is evidence that our duality is compatible with these functions. Indeed in the early real version of the theory this is proved in [GG]. However, it is not easy to see how duality of hypergeometric functions would go in the complex case. Nevertheless, there are reasons to believe that there should be such a duality. To begin with, the duality map preserves the GGMS stratifications, see [GGMS]. We recall the definition. The GGMS strata are parametrized by matroids on the set $1, 2, ..., n$. Two points x and y are in the same stratum if $M(x) = M(y)$. For M a matroid on $1, 2, ..., n$ we let S_M denote the corresponding stratum. The dual M^* of a matroid M is defined and discussed in [Ox], Chapter 2. We have the following theorem [HM]

Theorem 9.1.
$$\Psi(S_M) = S_{M^*}.$$

However it is not clear that the duality map lifts to the families of arrangement complements over the strata that give rise to the hypergeometric functions.

If a duality of hypergeometric functions could be established it would afford the opportunity to carry over the very detailed information on monodromy obtained in [DM] to some of the Gelfand examples.

9.3. **Self-dual torus orbits.** A very concrete problem suggested by the work of [DO] is the problem of finding the fixed points of the duality map ("self-associated point sets" in the terminology of [DO]). This problem is discussed in detail in Chapter III of [DO]. We note that in [Foth], P. Foth gave a description of the fixed-point set of an *anti-holomorphic* involution on a weight variety. The problem of finding the self-dual torus orbits of flags is the analogous problem for the (holomorphic) Chevalley involution. However the results of [DO] suggest this problem will be more subtle. For example, it is proved in [DO] following [Co] that in two cases the fixed set is closely related to the moduli theory of curves. Since one of these results is very easy to describe we conclude with it. Note first that we obtain a self-dual torus quotient by giving $Gr_n(\mathbb{C}^{2n})$ the symplectic form corresponding to $2\varpi_n$ and taking $\mathbf{r} = (1, 1, \cdots, 1) = \varpi_{2n}$ so $a = 2 = |\mathbf{r}|/n$. In [DO], Theorem 4, pg. 51, it is proved

that the fixed set S_{n-1} of the self-duality $\overline{\Theta}$ acting on the resulting torus quotient (equivalently the moduli space of $2n$–tuples of equally weighted (by 1) points in $\mathbb{CP}^{n-1}$) is a rational subvariety of dimension $\frac{n(n+1)}{2}$. In Example 4, pg. 37, of [DO] it is proved that S_2 is isomorphic to the Baily-Borel-Satake compactification of the (level two) Siegel modular variety of genus 2.

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